Topology

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General Topology

Chapter 1

Topological Spaces

1.1 Topological Spaces

Definition 1.1. One calls **topological space** a set X equipped with a family $\mathscr{U}$ of subsets of X, called the **open** sets of X, satisfying the following conditions:

(i) the subsets $\varnothing$ and X of X are open,

(ii) every union of open subsets of X is open,

(iii) every finite intersection of open subsets of X is open.

One says that $\mathscr{U}$ defines a **topology** on X.

Example. Consider a set X. The collection of all subsets of X is a topology on X, and is called the **discrete topology** on X. The collection consisting of X and $\varnothing$ is also a topology, and is called the **trivial topology** on X.

Example. Consider a set X. Let $\mathscr{U}_f$ be the collection of all subsets A of X such that $X \setminus A$ is either finite or is X. Then, $\mathscr{U}_f$ is a topology called the **finite complement topology** on X. Both X and $\varnothing$ are in $\mathscr{U}_f$, since $X \setminus X = \varnothing$ is finite and $X \setminus \varnothing = X$. If $\{A_i\}_{i\in I}$ is a family of nonempty elements of $\mathscr{U}_f$, since $X \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I}(X \setminus A_i)$ is finite, then $\bigcup_{i\in I} A_i \in \mathscr{U}_f$. In case I is finite, $X \setminus \bigcap_{i\in I} A_i = \bigcup_{i\in I}(X \setminus A_i)$ is consequently finite, then $\bigcap_{i\in I} A_i \in \mathscr{U}_f$.

Definition 1.2. Let X be a topological space, and $A \subseteq X$. One says that A is **closed** if $X \setminus A$ is open.

Proposition 1.3. *Let X be a topological space:*

(i) *the subsets $\varnothing$ and X of X are closed,*

(ii) *every intersection of closed subsets of X is closed,*

(iii) *every finite union of closed subsets of X is closed.*

Proof. The subsets $\varnothing$ and X are evidently closed by passage to complements. Let $\mathscr{C}$ a family of closed subsets of X. Since $X \setminus \bigcap_{B\in\mathscr{C}} B = \bigcup_{B\in\mathscr{C}} (X \setminus B)$ and $X \setminus B$ is open, then $X \setminus \bigcap_{B\in\mathscr{C}} B$ is open and $\bigcap_{B\in\mathscr{C}} B$

is consequently closed. If the family $\mathscr{C}$ is finite, since $X \setminus \bigcup_{B \in \mathscr{C}} B = \bigcap_{B \in \mathscr{C}} (X \setminus B)$ and $\bigcap_{B \in \mathscr{C}} (X \setminus B)$ is open, then $\bigcup_{B \in \mathscr{C}} B$ is closed. □

Definition 1.4. If X is a set, a **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X such that

(i) for each $x \in X$, there exists an element $B \in \mathscr{B}$ containing x,

(ii) if x belongs to the intersection of two elements $B_1, B_2 \in \mathscr{B}$, then there exists $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq B_1 \cap B_2$.

If $\mathscr{B}$ satisfies both conditions, then one defines the **topology generated** by $\mathscr{B}$ as follows: A subset U of X is said to be open in X if, for each $x \in U$, there exists $B \in \mathscr{B}$ such that $x \in B$ and $B \subseteq U$.

Proposition 1.5. *Let X be a set, and $\mathscr{B}$ a basis for a topology $\mathscr{U}$ on X. Then, $\mathscr{U}$ equals the collection formed by all unions of elements in $\mathscr{B}$.*

Proof. As $\mathscr{U}$ is a topology, any union of elements in $\mathscr{B}$ clearly belongs to $\mathscr{U}$. Conversely, given $U \in \mathscr{U}$, for each $x \in U$, there exists $B_x \in \mathscr{B}$ such that $x \in B_x$ and $B_x \subseteq U$ as $\mathscr{B}$ is a basis. So $\bigcup_{x \in U} B_x \subseteq U$, and we also have $U \subseteq \bigcup_{x \in U} B_x$ since $\bigcup_{x \in U} B_x$ contains every element of U. □

Proposition 1.6. *Let X be a set equipped with a topology $\mathscr{U}$. Suppose that $\mathscr{C}$ is a collection of open sets such that, for each $U \in \mathscr{U}$ and each $x \in U$, there exists $C \in \mathscr{C}$ such that $x \in C$ and $C \subseteq U$. Then, $\mathscr{C}$ is a basis for $\mathscr{U}$.*

Proof. We first prove that $\mathscr{C}$ is a basis. For the first condition, given $x \in X$, since $X \in \mathscr{U}$, then there exists $C \in \mathscr{C}$ such that $x \in C$ and $C \subseteq \mathscr{C}$. For the second condition, let $x \in C_1 \cap C_2$ where $C_1, C_2 \in \mathscr{C}$. Since C_1 and C_2 are open, so is $C_1 \cap C_2$, then there exists $C \in \mathscr{C}$ such that $x \in C$ and $C \subseteq C_1 \cap C_2$.

We now prove that the topology $\mathscr{T}$ generated by $\mathscr{C}$ is $\mathscr{U}$. If $U \in \mathscr{U}$ and $x \in U$, there exists $C \in \mathscr{C}$ such that $x \in C$ and $C \subseteq U$, and consequently $U \in \mathscr{T}$ by definition. Conversely, if $T \in \mathscr{T}$, then T equals a union of elements in $\mathscr{C}$ from Proposition 1.5. As $\mathscr{C} \subseteq \mathscr{U}$ and $\mathscr{U}$ is a topology, then $T \in \mathscr{U}$. □

1.2 Neighborhoods

Definition 1.7. Let X be a topological space, and $x \in X$. A subset V of X is called a **neighborhood** of x in X if there exists an open subset A of X such that $x \in A$ and $A \subseteq V$.

Proposition 1.8. *Let X be a topological space, and $x \in X$.*

(i) *If V and V' are neighborhoods of x, then $V \cap V'$ is a neighborhood of x.*

(ii) *If V is a neighborhood of x, and W a subset such that $V \subseteq W$, then W is a neighborhood of x.*

Proof. There exists open subsets U, U' containing x such that $U \subseteq V$ and $U' \subseteq V'$. So, $U \cap U'$ is an open subset of X containing x with the property $U \cap U' \subseteq V \cap V'$. If $V \subseteq W$, then $U \subseteq W$, and W is obviously a neighborhood of x. □

Proposition 1.9. *Let X be a topological space, and $A \subseteq X$. These conditions are equivalent:*

(i) *A is open,*

(ii) *A is a neighborhood of each of its points.*

Proof. $(i) \Rightarrow (ii)$: For a point x of A, we obviously have $x \in A \subseteq A$, so A is a neighborhood of x.

$(ii) \Rightarrow (i)$: For every $x \in A$, there exists an open subset A_x of X containing x such that that $A_x \subseteq A$. Then, the union $\bigcup_{x \in A} A_x$ is open, and is included in A. Since each point of A is contained in $\bigcup_{x \in A} A_x$, then $A \subseteq \bigcup_{x \in A} A_x$. Thus $A = \bigcup_{x \in A} A_x$, and A is consequently open. □

Definition 1.10. Let X be a topological space, and $x \in X$. One calls **fundamental system of neighborhoods** of x any family $\{V_i\}_{i \in I}$ of neighborhoods of x such that every neighborhood of x contains one of the V_i.

Example. Let X be a topological space, and $x \in X$. The set of all open subsets of X containing x is a fundamental system of neighborhoods of x.

1.3 Interior

Definition 1.11. Let X be a topological space, $A \subseteq X$, and $x \in X$. The point x is **interior** to A if A is a neighborhood of x in X. The set of all points interior to A is called the interior of A and denoted A°.

Proposition 1.12. *Let X be a topological space, and A a subset of X. Then A° is the largest open set of X contained in A.*

Proof. Let U be an open subset of X contained in A. If $x \in U$, then A is neighborhood of x, therefore $x \in A^\circ$, and consequently $U \subseteq A^\circ$. So, every open subset contained in A is included in A°.
Now, if $x \in A^\circ$, there exists an open subset B such that $x \in B$ and $B \subseteq A$. Then $B \subseteq A^\circ$ by the first part of the proof, thus A° is a neighborhood of x. From Proposition 1.9, we deduce that A° is open. □

Proposition 1.13. *Let X be a topological space, and $A \subseteq X$. These conditions are equivalent:*

(i) *A is open,*

(ii) $A = A^\circ$.

Proof. $(i) \Rightarrow (ii)$: If A is open, then $A = A^\circ$ from Proposition 1.12.

$(ii) \Rightarrow (i)$: If $A = A^\circ$, then A is open since A° is open. □

Proposition 1.14. *Let X be a topological space, and $A, B \subseteq X$. Then $(A \cap B)^\circ = A^\circ \cap B^\circ$.*

Proof. It is clear that $(A \cap B)^\circ \subseteq A^\circ$ and $(A \cap B)^\circ \subseteq B^\circ$, hence $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.
One has $A^\circ \subseteq A$ and $B^\circ \subseteq B$, therefore $A^\circ \cap B^\circ \subseteq A \cap B$. Since $A^\circ \cap B^\circ$ is open, then $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$ from Proposition 1.12. □

Definition 1.15. Let X be a topological space, and $A \subseteq X$. The **boundary** of A is the closed set $\partial A := X \setminus \big(A^\circ \sqcup (X \setminus A)^\circ\big)$.

1.4 Closure

Definition 1.16. Let X be a topological space, $A \subseteq X$, and $x \in X$. One says that x is **adherent** to A if every neighborhood of x in X intersects A. The set of all points adherent to A is called the **closure** of A and denoted $\overline{A}$.

Proposition 1.17. *Let X be a topological space, and $A \subseteq X$. Then $\overline{A} = X \setminus (X \setminus A)^\circ$.*

Proof. Take a point $x \in X$. We have $x \notin \overline{A}$ if and only if x has a neighborhood disjoint from A if and only if $x \in (X \setminus A)^\circ$. □

Proposition 1.18. *Let X be a topological space, and $A, B \subseteq X$.*

(i) $\overline{A}$ is the smallest closed subset of X containing A.

(ii) A is closed if and only if $A = \overline{A}$.

(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (i) : The interior $(X \setminus A)^\circ$ is the largest open set contained in $X \setminus A$. Therefore its complement $\overline{A}$ is closed and contains A. If B is a closed subset of X containing A, then $X \setminus B \subseteq (X \setminus A)^\circ = X \setminus \overline{A}$, and $\overline{A} \subseteq B$.
(ii) : As $\overline{A}$ is the smallest closed subset of X containing A, then A is closed if and only if $A = \overline{A}$.
(iii) : From Proposition 1.17, we have $\overline{A \cup B} = X \setminus \big(X \setminus (A \cup B)\big)^\circ = X \setminus \big((X \setminus A) \cap (X \setminus B)\big)^\circ$. Using Proposition 1.14, then $\overline{A \cup B} = X \setminus \big((X \setminus A)^\circ \cap (X \setminus B)^\circ\big) = \big(X \setminus (X \setminus A)^\circ\big) \cup \big(X \setminus (X \setminus B)^\circ\big) = \overline{A} \cup \overline{B}$. □

Definition 1.19. Let X be a topological space, and $A \subseteq X$. One says A is **dense** if $\overline{A} = X$.

Proposition 1.20. *Let X be a topological space, and $A \subseteq X$. These conditions are equivalent:*

(i) A is dense,

(ii) $(X \setminus A)^\circ = \varnothing$,

(iii) every nonempty open subset of X intersects A.

Proof. $(i) \Rightarrow (ii)$: Since $X \setminus (X \setminus A)^\circ = \overline{A} = X$, then $(X \setminus A)^\circ = \varnothing$.
$(ii) \Rightarrow (iii)$: Let U be an open subset that does not intersect A. Therefore $U \subseteq (X \setminus A)^\circ = \varnothing$.
$(iii) \Rightarrow (i)$: Since every neighborhood of every point of X intersects A, then $\overline{A} = X$. □

1.5 Separated Topological Spaces

Definition 1.21. A topological space X is said to be **separated** if any two distinct points of X admit disjoint neighborhoods.

Proposition 1.22. *Let X be a separated topological space, and $x \in X$. Then $\{x\}$ is closed.*

Proof. Take a point $y \in X \setminus \{x\}$. There exist neighborhoods V and W of x and y respectively that are disjoint. In particular, $W \subseteq X \setminus \{x\}$, hence $X \setminus \{x\}$ is neighborhood of y. Thus $X \setminus \{x\}$ is a neighborhood of each of its points. We deduce from Proposition 1.9 that $X \setminus \{x\}$ is open. □

Chapter 2

Limit and Continuity

2.1 Limits

Definition 2.1. A **filter** on a set X is a set $\mathscr{F}$ formed by nonempty subsets of X satisfying the following conditions:

(i) if $A \in \mathscr{F}$ and $B \in \mathscr{F}$, then $A \cap B \in \mathscr{F}$,

(ii) if $A \in \mathscr{F}$ and if A' is a subset of X containing A, then $A' \in \mathscr{F}$.

Definition 2.2. A **filter base** on a set X is a set $\mathscr{B}$ of nonempty subsets of X such that, if $A \in \mathscr{B}$ and $B \in \mathscr{B}$, there exists $C \in \mathscr{B}$ such that $C \subseteq A \cap B$.

Example. Let X be a topological space, and $x_0 \in X$. The set $\mathscr{V}$ formed by the neighborhoods of x_0 is a filter on X. A fundamental system of neighborhoods of x_0 is a filter base on X. Let $Y \subseteq X$, and assume $x_0 \in \overline{Y}$. The set $\{Y \cap V \mid V \in \mathscr{V}\}$ is a filter on Y.

Example. For $x \in \mathbb{R}$, the set of intervals $\big\{(x-\varepsilon, x+\varepsilon)\big\}_{\varepsilon \in \mathbb{R}_+^*}$ is a filter base on $\mathbb{R}$.

Definition 2.3. Let X be a set equipped with a filter base $\mathscr{B}$, Y a topological space, $f: X \to Y$ a function, and l a point of Y. One says that f tends to l along $\mathscr{B}$ if, for every neighborhood V of l in Y, there exists $B \in \mathscr{B}$ such that $f(B) \subseteq V$.
If X is a topological space, and $\mathscr{B}$ the filter formed by the neighborhoods of a point x_0 of X, one says that l is the **limit** of f along the neighborhood filter of x_0, and writes $\lim\limits_{x \to x_0} f(x) = l$.

Proposition 2.4. *Let X, Y be topological spaces, $f: X \to Y$ a function, $x_0 \in X$, $l \in Y$, $\{V_i\}_{i \in I}$ a fundamental system of neighborhoods of x_0 in X, and $\{W_j\}_{j \in J}$ a fundamental system of neighborhoods of l in Y. The following conditions are equivalent:*

(i) $\lim\limits_{x \to x_0} f(x) = l,$

(ii) *for every $j \in J$, there exists $i \in I$ such that $f(V_i) \subseteq W_j$.*

Proof. $(i) \Rightarrow (ii)$: For every $j \in J$, there exists a neighborhood V of x_0 such that $f(V) \subseteq W_j$. By definition, there exists $i \in I$ such that $V_i \subseteq V$. Therefore $f(V_i) \subseteq W_j$.
$(ii) \Rightarrow (i)$: Let W be a neighborhood of l. There exists $j \in J$ such that $W_i \subseteq W$. Then, there exists $i \in I$ such that $f(V_i) \subseteq W_j$, and consequently $f(V_i) \subseteq W$. □

Proposition 2.5. *Let X be a set equipped with a filter base $\mathscr{B}$, Y a separated topological space, and $f : X \to Y$ a function. If f admits a limit along $\mathscr{B}$, this limit is unique.*

Proof. Let l, l' be distinct limits of f along $\mathscr{B}$. Since Y is separated, there exist disjoint neighborhoods V and V' of l and l' respectively in Y. There exist $B, B' \in \mathscr{B}$ such that $f(B) \subseteq V$ and $f(B') \subseteq V'$. By definition, there exists $B'' \in \mathscr{B}$ such that $B'' \subseteq B \cap B'$. Then $f(B'') \subseteq f(B) \cap f(B') \subseteq V \cap V'$. Since B'' is nonempty, then $f(B'') \neq \varnothing$, and consequently $V \cap V' \neq \varnothing$ which is absurd. □

Proposition 2.6. *Let X be a set equipped with a filter base $\mathscr{B}$, Y a topological space, $f : X \to Y$ a function, and $l \in Y$. Let $X' \in \mathscr{B}$, and f' the restriction of f to X'. The sets $B \cap X'$, where $B \in \mathscr{B}$, form a filter base $\mathscr{B}'$ on X'. The following conditions are equivalent:*

(i) *f tends to l along $\mathscr{B}$,*

(ii) *f' tends to l along $\mathscr{B}'$.*

Proof. $(i) \Rightarrow (ii)$: Let V be a neighborhood of l. There exists $B \in \mathscr{B}$ such that $f(B) \subseteq V$. Hence $f'(B \cap X') \subseteq V$. As $B \cap X' \mathscr{B}'$, then f' tends to l along $\mathscr{B}'$.
$(ii) \Rightarrow (i)$: Let V be a neighborhood of l. There exists $B' \in \mathscr{B}'$ such that $f(B') \subseteq V$. But B' has the form $B \cap X'$ with $B \in \mathscr{B}$. Since $X' \in \mathscr{B}$, there exists $B'' \in \mathscr{B}$ such that $B'' \subseteq B \cap X'$. Then, $f(B'') \subseteq f'(B') \subseteq V$, and f consequently tends to l along $\mathscr{B}$. □

2.2 Adherence Values

Definition 2.7. Let X be a set equipped with a filter base $\mathscr{B}$, Y a topological space, $f : X \to Y$ a function, and l a point of Y. One says that l is an **adherence value** of f along $\mathscr{B}$ if, for every neighborhood V of l and for every $B \in \mathscr{B}$, $f(B)$ intersects V.

Example. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \{x\}$. Then, every real number in $[0,1)$ is an adherence value of f along the filter base $\big\{(a,+\infty)\big\}_{a \in \mathbb{R}_+}$.

Proposition 2.8. *Let X be a set equipped with a filter base $\mathscr{B}$, Y a separated topological space, $f : X \to Y$ a function, and l a point of Y. If f tends to l along $\mathscr{B}$, then l is the unique adherence value of f along $\mathscr{B}$.*

Proof. Let V be a neighborhood of l, and $B \in \mathscr{B}$. There exists $B' \in \mathscr{B}$ such that $f(B') \subseteq V$. Then $B \cap B' \neq \varnothing$, hence $f(B \cap B') \neq \varnothing$, and $f(B \cap B') \subseteq f(B) \cap V$. Therefore $f(B)$ intersects V, meaning that l is an adherence value of f along $\mathscr{B}$.
Let l' be an adherence value of f along $\mathscr{B}$, assume $l' \neq l$. There exist neighborhoods V and V' of l and l' respectively that are disjoint. There exists $B \in \mathscr{B}$ such that $f(B) \subseteq V$. Then $f(B) \cap V'$ contradicting the fact that l' is an adherence value. □

Proposition 2.9. *Let X be a set equipped with a filter base $\mathscr{B}$, Y a topological space, and $f : X \to Y$ a function. The set formed by the adherence values of f along $\mathscr{B}$ is $\bigcap_{B \in \mathscr{B}} \overline{f(B)}$.*

Proof. Let l be an adherence value of f along $\mathscr{B}$, and $B \in \mathscr{B}$. Every neighborhood of l intersects $f(B)$. Then $l \in \overline{f(B)}$, and $l \in \bigcap_{B \in \mathscr{B}} \overline{f(B)}$.

Let $l' \in \bigcap_{B \in \mathscr{B}} \overline{f(B)}$, V' be a neighborhood of l', and take $B \in \mathscr{B}$. Since $l' \in \overline{f(B)}$, then $f(B)$ intersects V', and l' is an adherence value of f. □

2.3 Continuity

Definition 2.10. Let X,Y be topological spaces, $f : X \to Y$ a function, and $x_0 \in X$. One says that f is **continuous** at x_0 if $\lim_{x \to x_0} f(x) = f(x_0)$. In other words, for every neighborhood V of $f(x_0)$, there exists a neighborhood U of x_0 such that $f(U) \subseteq V$.

Proposition 2.11. *Let X,Y,Z be topological spaces, $f : X \to Y$ and $g : Y \to Z$ functions, and $x_0 \in X$. If f is continuous at x_0, and g at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Proof. Let W be a neighborhood of $g\big(f(x_0)\big)$ in Z. There exists a neighborhood V of $f(x_0)$ in Y such that $g(V) \subseteq W$. Moreover, there exists a neighborhood U of x_0 in X such that $f(U) \subseteq V$. Then, U is neighborhood of U such that $g \circ f(U) \subseteq g(V) \subseteq W$. □

Definition 2.12. Let X,Y be topological spaces, and $f : X \to Y$ a function. One says that f is continuous on X if f is continuous at every point of X. The set of continuous functions from X into Y is denoted $\mathscr{C}(X,Y)$.

Example. Let $A,B \subseteq \mathbb{R}^n$, and f a rational function such that f is defined on A and $f(A) = B$. Consider the basis $\mathscr{B}_A = \big\{A \cap \mathbb{B}(x,r) \,\big|\, x \in A,\, r \in \mathbb{R}_+^*\big\}$ resp. $\mathscr{B}_B = \big\{B \cap \mathbb{B}(x,r) \,\big|\, x \in B,\, r \in \mathbb{R}_+^*\big\}$ for a topology on A resp. B, where $\mathbb{B}(x,r)$ is the open n-ball $\big\{y \in \mathbb{R}^n \,\big|\, \|x-y\|_2 < r\big\}$. Take $x_0 \in A$, and a neighborhood V of $f(x_0)$. There exists an open ball $\mathbb{B}(x_0,r)$ such that $A \cap \mathbb{B}(x_0,r) \subseteq f^{-1}(V)$. So $f\big(A \cap \mathbb{B}(x_0,r)\big) \subseteq V$, and $f : A \to B$ is consequently continuous.

Proposition 2.13. *Let X,Y,Z be topological spaces, $f \in \mathscr{C}(X,Y)$, and $g \in \mathscr{C}(Y,Z)$. Then, we have $g \circ f \in \mathscr{C}(X,Z)$.*

Proof. Use Proposition 2.11 for the continuity of $g \circ f$ on every point of X. □

Proposition 2.14. *Let X,Y be topological spaces, and $f : X \to Y$ a function. The following conditions are equivalent:*

- (i) *f is continuous,*
- (ii) *$f^{-1}(B)$ is an open subset of X if B is an open subset of Y,*
- (iii) *$f^{-1}(B)$ is a closed subset of X if B is a closed subset of Y,*
- (iv) *for every subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$.*

Proof. $(i) \Rightarrow (iv)$: Let $A \subseteq X$ and $x_0 \in \overline{A}$. Take a neighborhood W of $f(x_0)$ in Y. Since f is continuous at x_0, there exists a neighborhood V of x_0 in X such that $f(V) \subseteq W$. The fact $x_0 \in \overline{A}$ implies $V \cap A \neq \varnothing$. As $f(V \cap A) \subseteq W \cap f(A)$, one sees that $W \cap f(A) \neq \varnothing$. Therefore $f(x_0) \in \overline{f(A)}$, and $f(\overline{A}) \subseteq \overline{f(A)}$.

$(iv) \Rightarrow (iii)$: Let B be a closed subset of Y, and $A \in f^{-1}(B)$. Then $f(A) \subseteq B$, and $\overline{f(A)} \subseteq B$ from Proposition 1.18 (i). If $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$ as f is continuous. Therefore $f(x) \in B$ and so $x \in A$. Thus $A = \overline{A}$.

$(iii) \Rightarrow (ii)$: Let B be an open subset of Y. Then $Y \setminus B$ is closed, and consequently $f^{-1}(Y \setminus B)$ is closed. But $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$, then $f^{-1}(B)$ is open.

$(ii) \Rightarrow (i)$: Let $x_0 \in X$, and W a neighborhood of $f(x_0)$ in Y. There exists an open subset B of Y such that $f(x_0) \in B \subseteq W$. If $A = f^{-1}(B)$, then A is open, and A is a neighborhood of x_0 as $x_0 \in A$. Since $f(A) \subseteq B \subseteq W$, then f is continuous at x_0. □

2.4 Homeomorphisms

Proposition 2.15. *Let X,Y be topological spaces, and $f : X \to Y$ a bijective function. The following conditions are equivalent:*

(i) *f and f^{-1} are a continuous,*

(ii) *a subset A of X is open if and only if $f(A)$ is open in Y,*

(iii) *a subset A of X is closed if and only if $f(A)$ is closed in Y.*

Proof. $(i) \Rightarrow (ii)$: Using Proposition 2.14, we deduce from the continuity of f that if $f(A)$ is open then A is open, and from the continuity of f^{-1} that if A is open then $f(A)$ is open. One analogously proves $(i) \Rightarrow (iii)$.

$(ii) \Rightarrow (i)$: Using Proposition 2.14, "if $f(A)$ is open then A is open" implies that f is continuous, and "if A is open then $f(A)$ is open" implies that f^{-1} is continuous. One analogously gets $(iii) \Rightarrow (i)$. □

Definition 2.16. Let X,Y be topological spaces, and f a function from X into Y. One says that f is a **homeomorphism** if f is bijective, continuous, and f^{-1} is continuous. In that case, one says that X and Y are homeomorphic.

Example. The n-dimensional sphere is the set $\mathbb{S}^n := \big\{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} \bigm| x_1^2+\dots+x_{n+1}^2 = 1\big\}$. Let $a = (0,\dots,0,1) \in \mathbb{S}^n$, and identify $\mathbb{R}^n$ with $\big\{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{n+1} \bigm| x_{n+1} = 0\big\}$. We are going to define a homeomorphism from $\mathbb{S}^n \setminus \{a\}$ onto $\mathbb{R}^n$. Take a point $x = (x_1,\dots,x_{n+1}) \in \mathbb{S}^n \setminus \{a\}$. The line joining a and x is $D = \Big\{\big(\lambda x_1,\dots,\lambda x_n, 1+\lambda(x_{n+1}-1)\big) \in \mathbb{R}^{n+1} \Bigm| \lambda \in \mathbb{R}\Big\}$. This line touches $\mathbb{R}^n$ when $1+\lambda(x_{n+1}-1) = 0$, that is when $\lambda = \dfrac{1}{1-x_{n+1}}$. Thus $D \cap \mathbb{R}^n$ reduces to the point $f(x)$ with coordinates

$$x'_1 = \frac{x_1}{1-x_{n+1}}, \quad x'_2 = \frac{x_2}{1-x_{n+1}}, \ \dots\ , \quad x'_n = \frac{x_n}{1-x_{n+1}}, \quad x'_{n+1} = 0. \tag{2.1}$$

We have thus defined a function $f : \mathbb{S}^n \setminus \{a\} \to \mathbb{R}^n$. We now prove that, given $x' = (x'_1,\dots,x'_n,0)$, there exists one and only one point $x = (x_1,\dots,x_{n+1})$ in $\mathbb{S}^n \setminus \{a\}$ such that $f(x) = x'$. The solution of Equation 2.1 yields the conditions

$$x_i = x'_i(1-x_{n+1}) \text{ for } 1 \leq i \leq n, \quad \text{and} \quad \sum_{i=1}^{n} x_i'^2(1-x_{n+1})^2 + x_{n+1}^2 = 1.$$

After dividing out $1-x_{n+1}$, we obtain $({x'_1}^2+\dots+{x'_n}^2)(1-x_{n+1}) - 1 - x_{n+1} = 0$, which gives

$$x_{n+1} = \frac{{x'_1}^2+\dots+{x'_n}^2-1}{{x'_1}^2+\dots+{x'_n}^2+1} \quad \text{and} \quad x_1 = \frac{2x'_1}{{x'_1}^2+\dots+{x'_n}^2+1}, \ \dots\ , x_n = \frac{2x'_n}{{x'_1}^2+\dots+{x'_n}^2+1}. \tag{2.2}$$

Thus $f : \mathbb{S}^n \setminus \{a\} \to \mathbb{R}^n$ is a bijection. Let $\mathscr{B}_{\mathbb{S}^n\setminus\{a\}} = \big\{\mathbb{S}^n \setminus \{a\} \cap \mathbb{B}(x,r) \bigm| x \in \mathbb{S}^n \setminus \{a\}, r \in \mathbb{R}_+^*\big\}$ resp. $\mathscr{B}_{\mathbb{R}^n} = \big\{\mathbb{R}^n \cap \mathbb{B}(x,r) \bigm| x \in \mathbb{R}^n, r \in \mathbb{R}_+^*\big\}$ be a basis for a topology on $\mathbb{S}^n \setminus \{a\}$ resp. $\mathbb{R}^n$, where $\mathbb{B}(x,r)$ is the open $n+1$-ball $\big\{y \in \mathbb{R}^{n+1} \bigm| \|x-y\|_2 < r\big\}$. We see in Equation 2.1 resp. Equation 2.2 that f resp. f^{-1} is a rational function, and is consequently continuous. Hence f is a homeomorphism called stereographic projection of $\mathbb{S}^n \setminus \{a\}$ onto $\mathbb{R}^n$.

Chapter 3

Construction of Topological Spaces

3.1 Topological Subspaces

Proposition 3.1. *Let X be a topological space, $\mathscr{U}$ a topology on X, and Y a subset of X. Then $\mathscr{V} = \{U \cap Y \mid U \in \mathscr{U}\}$ is a topology on Y.*

Proof. (i) : As $\varnothing, X \in \mathscr{U}$, then $\varnothing = \varnothing \cap Y \in \mathscr{V}$ and $Y = X \cap Y \in \mathscr{V}$.
(ii) : Let $\{V_i\}_{i\in I}$ be a family of subsets belonging to $\mathscr{V}$. For every $i \in I$, there exists $U_i \in \mathscr{U}$ such that $V_i = U_i \cap Y$. Therefore $\bigcup_{i\in I} V_i = \bigcup_{i\in I}(U_i \cap Y) = \Big(\bigcup_{i\in I} U_i\Big) \cap Y \in \mathscr{V}$.
(iii) : If I is finite, then $\bigcap_{i\in I} V_i = \bigcap_{i\in I}(U_i \cap Y) = \Big(\bigcap_{i\in I} U_i\Big) \cap Y \in \mathscr{V}$. □

Definition 3.2. Let X be a topological space, $\mathscr{U}$ a topology on X, and Y a subset of X. The set $\mathscr{V} = \{U \cap Y \mid U \in \mathscr{U}\}$ is called the **topology induced** on Y by the given topology of X. Equipped with this topology, Y is called a **topological subspace** of X.

Example. Consider $\mathbb{R}$ with the usual topology. As $\{n\} = \mathbb{Z} \cap \big(n - \frac{1}{2}, n + \frac{1}{2}\big)$, every point set $\{n\}$ of $\mathbb{Z}$ is therefore open. Every subset of $\mathbb{Z}$ is the union of point sets, then is open. Thus the topological subspace $\mathbb{Z}$ of $\mathbb{R}$ is discrete.

Proposition 3.3. *Let X be a topological space, Y a subspace of X, and A a subset of Y. The following conditions are equivalent:*

(i) *A is closed in Y,*

(ii) *A is the intersection with Y of a closed subset of X.*

Proof. $(i) \Rightarrow (ii)$: The subset $Y \setminus A$ is open in Y. Therefore there exists an open subset U of X such that $Y \setminus A = U \cap Y$. Thus $A = (X \setminus U) \cap Y$, and since $X \setminus U$ is closed, we get the result.
$(ii) \Rightarrow (i)$: Suppose $A = V \cap Y$ where V is closed subset of X. Then $Y \setminus A = (X \setminus V) \cap Y$. Since $X \setminus V$ is open in X, then $Y \setminus A$ is open in Y, and A is closed in Y. □

Proposition 3.4. *Let X be a topological space, Y a subspace of X, and $x \in Y$. For a subset A of Y, the following conditions are equivalent:*

(i) *A is a neighborhood of x in Y,*

(ii) *A is the intersection with Y of a neighborhood of x in X.*

Proof. $(i) \Rightarrow (ii)$: There exists an open subset B of Y such that $x \in B \subseteq A$. Then there exists an open subset U of X such that $B = U \cap Y$. Letting $V = U \cup A$, we have $x \in V$, thus V is a neighborhood of x in X. Besides, $Y \cap V = (Y \cap U) \cup (Y \cap A) = B \cup A = A$.

$(ii) \Rightarrow (i)$: Suppose $A = Y \cap V$ where V is a neighborhood of x in X. There exists an open subset U of X such that $x \in U \subseteq V$. Then $x \in Y \cap U \subseteq Y \subseteq V = A$, and since $Y \cap U$ is open in Y, thus A is neighborhood of x in Y. □

Proposition 3.5. *Let X be a topological space, and $Y \subseteq X$. If X is separated, then Y is separated.*

Proof. Take two distinct points x, y of Y. There exist disjoint neighborhoods U and V of x and y respectively in X. We deduce from Proposition 3.4 that $U \cap Y$ and $V \cap Y$ are neighborhoods of x and y respectively in Y, and they are disjoint. □

Proposition 3.6. *Let X, Y, Z be topological spaces such that $X \supseteq Y \supseteq Z$. Assume $\mathscr{U}$ is a topology on X, $\mathscr{V}$ the topology induced by $\mathscr{U}$ on Y, and $\mathscr{W}$ the topology induced by $\mathscr{V}$ on Z. Then $\mathscr{W}$ is the topology induced by $\mathscr{U}$ on Z.*

Proof. Let $\mathscr{W}'$ be the topology induced by $\mathscr{U}$ on Z.
For $W \in \mathscr{W}$, there exist $V \in \mathscr{V}$ such that $W = V \cap Z$, and $U \in \mathscr{U}$ such that $V = U \cap Y$. Then $W = U \cap Z$, and consequently $W \in \mathscr{W}'$.
For $W' \in \mathscr{W}'$, there exists $U \in \mathscr{U}$ such that $W' = U \cap Z$. If $V = U \cap Y$, then $V \in \mathscr{V}$ and $W' = V \cap Z$. Therefore $W' \in \mathscr{W}$. □

Proposition 3.7. *Let X be a set equipped with a filter base $\mathscr{B}$, Y a topological space, Y' a subspace of Y, $f : X \to Y'$ a function, and l a point of Y'. The following conditions are equivalent:*

(i) *f tends to l along $\mathscr{B}$ relative to Y',*

(ii) *f tends to l along $\mathscr{B}$ relative to Y.*

Proof. $(i) \Rightarrow (ii)$: Let V be a neighborhood of l in Y. We know from Proposition 3.4 that $V \cap Y'$ is a neighborhood of l in Y'. There exists $B \in \mathscr{B}$ such that $f(B) \subseteq V \cap Y'$. Thus $f(B) \subseteq V$, and f consequently tends to l along $\mathscr{B}$ relative to Y.

$(ii) \Rightarrow (i)$: Let V' be a neighborhood of l' in Y'. From Proposition 3.4, there exists a neighborhood V of l in Y such that $V \cap Y' = V'$. Besides, there exists $B \in \mathscr{B}$ such that $f(B) \subseteq V$. Since $f(X) \subseteq Y'$, one has $f(B) \subseteq V \cap Y'$ which is V'. Thus f tends to l along $\mathscr{B}$ relative to Y. □

Corollary 3.8. *Let X, Y be topological spaces, Y' a subspace of Y, and $f : X \to Y'$ a function. The following conditions are equivalent:*

(i) *f is continuous,*

(ii) *f, regarded as a function from X into Y, is continuous.*

Proof. For every $x_0 \in X$, the condition $\lim\limits_{x \to x_0} f(x) = f(x_0)$ has the same meaning, according to Proposition 3.7 for the neighborhood filter of x_0, whether one considers f to have values in Y' or in Y. □

3.2 Products of Topological Spaces

Proposition 3.9. *Let $X_1,\dots,X_n$ be topological spaces equipped with topologies $\mathscr{U}_1,\dots,\mathscr{U}_n$ respectively. The set $\mathscr{U}$ formed by any union of elements in $\mathscr{U}_1\times\cdots\times\mathscr{U}_n$ is a topology on $X=X_1\times\cdots\times X_n$.*

Proof. (i) : We have $X=X_1\times\cdots\times X_n\in\mathscr{U}_1\times\cdots\times\mathscr{U}_n$ and $\varnothing=\varnothing\times X_2\times\cdots\times X_n\in\mathscr{U}_1\times\cdots\times\mathscr{U}_n$.

(ii) : From its definition, any union of elements in $\mathscr{U}$ is a union of elements in $\mathscr{U}_1\times\cdots\times\mathscr{U}_n$.

(iii) : Take $A,B\in\mathscr{U}$. We have $A=\bigcup\limits_{\alpha\in I}A_\alpha$ and $B=\bigcup\limits_{\beta\in J}B_\beta$ with $A_\alpha,B_\beta\in\mathscr{U}_1\times\cdots\times\mathscr{U}_n$. Then $A\cap B=\bigcup\limits_{\substack{\alpha\in I\\ \beta\in J}}A_\alpha\cap B_\beta$. Setting $A_\alpha=A_1\times\cdots\times A_n$ and $B_\beta=B_1\times\cdots\times B_n$, we get

$$A_\alpha\cap B_\beta=(A_1\cap B_1)\times\cdots\times(A_n\cap B_n)\in\mathscr{U}_1\times\cdots\times\mathscr{U}_n.$$

□

Definition 3.10. Let $X_1,\dots,X_n$ be topological spaces equipped with topologies $\mathscr{U}_1,\dots,\mathscr{U}_n$ respectively. The topology $\mathscr{U}$ on $X=X_1\times\cdots\times X_n$ formed by any union of elements in $\mathscr{U}_1\times\cdots\times\mathscr{U}_n$ is called the **product topology** of the given topologies on $X_1,\dots,X_n$. Equipped with this topology, X is called the **product topological space** of the topological spaces $X_1,\dots,X_n$.

Proposition 3.11. *Let $X=X_1\times\cdots\times X_n$ be a product of topological spaces, and $x=(x_1,\dots,x_n)\in X$. The sets of the form $V_1\times\cdots\times V_n$, where V_i is a neighborhood of x_i in X_i, constitute a fundamental system of neighborhoods of x in X.*

Proof. For $i\in\{1,\dots,n\}$, let V_i be a neighborhood of x_i in X_i. There exists an open subset A_i of X_i such that $x_i\in A_i\subseteq V_i$. Then $x\in A_1\times\cdots\times A_n\subseteq V_1\times\cdots\times V_n$. As $A_1\times\cdots\times A_n$ is open in X, thus $V_1\times\cdots\times V_n$ is a neighborhood of x in X.
Let V be a neighborhood of x in X. There exists an open subset A of X such that $x\in A\subseteq V$. By definition of the product topology, there exists an open subset A_i such that $x_i\in A_i$ and $A_1\times\cdots\times A_n\subseteq A$. Thus A_i is a neighborhood of x_i and $A_1\times\cdots\times A_n\subseteq V$. □

Proposition 3.12. *Let $X=X_1\times\cdots\times X_n$ be a product of topological spaces. If each X_i is separated, then X is separated.*

Proof. Let $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$ be two distinct points of X. One has $x_i\neq y_i$ for at least one $i\in\{1,\dots,n\}$. If $x_1\neq y_1$ for example, there exist disjoint neighborhoods U and V of x_1 and y_1 respectively in X_1. Then $U\times X_2\times\cdots\times X_n$ and $V\times X_2\times\cdots\times X_n$ are disjoint neighborhoods of x and y respectively in X. □

Proposition 3.13. *Let X be a set equipped with a filter base $\mathscr{B}$, $Y=Y_1\times\cdots\times Y_n$ a product of topological spaces, and $l=(l_1,\dots,l_n)\in Y$. Consider a function $f:X\to Y$, that is, having the form $x\mapsto\big(f_1(x),\dots,f_n(x)\big)$, where $f_i:X\to Y_i$ is also a function for $i\in\{1,\dots,n\}$. Then, the following conditions are equivalent:*

(i) *f tends to l along $\mathscr{B}$,*

(ii) *f_i tends to l_i along $\mathscr{B}$.*

Proof. $(i) \Rightarrow (ii)$: Let us show, for example, that f_1 tends to l_1 along $\mathscr{B}$. If V_1 is a neighborhood of l_1, then $V_1 \times Y_2 \times \cdots \times Y_n$ is a neighborhood of l in Y. Therefore, there exists $B \in \mathscr{B}$ such that $f(B) \subseteq V_1 \times Y_2 \times \cdots \times Y_n$. Thus $f_1(B) \subseteq V$, and f_1 consequently tends to l_1 along $\mathscr{B}$.

$(ii) \Rightarrow (i)$: Let V be a neighborhood of l in Y. We know from Proposition 3.11 that there exist neighborhoods $V_1, \dots, V_n$ of $l_1, \dots, l_n$ respectively in $Y_1, \dots, Y_n$ such that $V_1 \times \cdots \times V_n \subseteq V$. Then, there exist $B_1, \dots, B_n \in \mathscr{B}$ such that $f_1(B_1) \subseteq V_1, \dots, f_n(B_n) \subseteq V_n$. Moreover, there exists $B \in \mathscr{B}$ such that $B \subseteq B_1 \cap \cdots \cap B_n$. Then, $f(B) \subseteq f_1(B_1) \times \cdots \times f_n(B_n) \subseteq V_1 \times \cdots \times V_n \subseteq V$, and f consequently tends to l along $\mathscr{B}$. □

Proposition 3.14. *Let X be a topological space, and $Y = Y_1 \times \cdots \times Y_n$ a product of topological spaces. Consider a function $f : X \to Y$, that is, having the form $x \mapsto \big(f_1(x), \dots, f_n(x)\big)$, where $f_i : X \to Y_i$ is also a function for $i \in \{1, \dots, n\}$. The following conditions are equivalent:*

- (i) *f is continuous,*
- (ii) *$f_1, \dots, f_n$ are continuous.*

Proof. For every $x_0 \in X$, the conditions $\lim\limits_{x \to x_0} f(x) = f(x_0)$ and $\lim\limits_{x \to x_0} f_i(x) = f_i(x_0)$, for $i \in \{1, \dots, n\}$, are equivalent by Proposition 3.13 using the neighborhood filter of x_0. □

3.3 Quotient Spaces

Proposition 3.15. *Let X be a topological space with topology $\mathscr{U}$, $\mathscr{R}$ an equivalence relation on X, and c the canonical mapping from X onto $X/\mathscr{R}$. Then the set defined by $\mathscr{V} := \big\{A \subseteq X/\mathscr{R} \,\big|\, c^{-1}(A) \in \mathscr{U}\big\}$ a topology on $X/\mathscr{R}$.*

Proof. The set $\varnothing$ and $X/\mathscr{R}$ are open in $X/\mathscr{R}$ since $c^{-1}(\varnothing) = \varnothing$ and $c^{-1}(X/\mathscr{R}) = X$. The two other conditions follow, for a set $\{A_i\}_{i \in I}$ included in $\mathscr{V}$, from the equations

$$c^{-1}\Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{i \in I} c^{-1}(A_i) \quad \text{and} \quad c^{-1}\Big(\bigcap_{i=1}^{n} A_i\Big) = \bigcap_{i=1}^{n} c^{-1}(A_i).$$

□

Definition 3.16. Let X be a topological space with topology $\mathscr{U}$, $\mathscr{R}$ an equivalence relation on X, and c the canonical mapping from X onto $X/\mathscr{R}$. The topology $\big\{A \subseteq X/\mathscr{R} \,\big|\, c^{-1}(A) \in \mathscr{U}\big\}$ on $X/\mathscr{R}$ is called the **quotient topology** of the topology of X by $\mathscr{R}$. Equipped with this topology, $X/\mathscr{R}$ is called the **quotient space** of X by $\mathscr{R}$.

Proposition 3.17. *Let X be a topological space, $\mathscr{R}$ an equivalence relation on X, c the canonical mapping from X onto $X/\mathscr{R}$, Y a topological space, and $f : X/\mathscr{R} \to Y$ a function. The following conditions are equivalent:*

- (i) *f is continuous on $X/\mathscr{R}$,*
- (ii) *the function $f \circ c : X \to Y$ is continuous.*

Proof. $(i) \Rightarrow (ii)$: The mapping c is continuous as, if A is open in $X/\mathscr{R}$, then $c^{-1}(A)$ is open in X. Since f is also continuous, then $f \circ c$ is continuous.

$(ii) \Rightarrow (i)$: Let B be an open subset of Y. Then $c^{-1}\big(f^{-1}(B)\big) = (f \circ c)^{-1}(B)$ is open in X. Therefore $f^{-1}(B)$ is open in $X/\mathscr{R}$ by the definition of c. Thus f is continuous from Proposition 2.14. □

Chapter 4

Compact Spaces

4.1 Compact Spaces

Definition 4.1. Let X be a set, and A a subset of X. A family $\mathscr{F}$ of subsets included in X is a **covering** of A if $A \subseteq \bigcup_{U \in \mathscr{F}} U$.

Definition 4.2. A topological space X is **compact** if, for any family $\mathscr{O}$ of open subsets of X covering X, one can extract from $\mathscr{O}$ a finite subfamily that again covers X. By passage to complements, this definition is equivalent, for any family $\mathscr{C}$ of closed subsets of X having empty intersection, to the existence of a finite subfamily of $\mathscr{C}$ having empty intersection.

Proposition 4.3. *Let X be a topological space, and A a subspace of X. The following conditions are equivalent:*

(i) *A is compact,*

(ii) *if a family of open subsets of X covers A, one can extract from it a finite subfamily that again covers A.*

Proof. $(i) \Rightarrow (ii)$: Let $\{U_i\}_{i\in I}$ be a family of open subsets of X such that $A \subseteq \bigcup_{i\in I} U_i$. Every $U_i \cap A$ is open in A, and the family $\{U_i \cap A\}_{i\in I}$ covers A, so there exists a finite subset J of I such that $A = \bigcup_{j\in J}(U_j \cap A)$. The subfamily $\{U_j\}_{j\in J}$ consequently covers A.

$(ii) \Rightarrow (i)$: Let $\{V_i\}_{i\in I}$ be a family of open sets of A covering A. For every $i \in I$, there exists an open subset U_i of X such that $V_i = U_i \cap A$. Then $\{U_i\}_{i\in I}$ covers A, there consequently exists a finite subset J of I such that $\{U_j\}_{j\in J}$ covers A. Therefore $\bigcup_{j\in J} V_j = A$. □

Theorem 4.4 (Borel-Lebesgue)**.** *Consider the space $\mathbb{R}$ equipped with the usual topology, and let $a, b \in \mathbb{R}$ with $a \leq b$. Then the interval $[a,b]$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be a family of open subsets of $\mathbb{R}$ covering $[a,b]$, and A be the set of $x \in [a,b]$ such that $[a,x]$ is covered by a finite subfamily of $\{U_i\}_{i\in I}$. The set A is nonempty since $a \in A$. It is contained in $[a,b]$, and therefore has a supremum m in $[a,b]$. There exists $j \in I$ such that $m \in U_j$. Since U_j is open in $\mathbb{R}$, there exists $\varepsilon > 0$ such that $[m-\varepsilon, m+\varepsilon] \subseteq U_j$. As m is the supremum of A, there exists $x \in A$ such that $m - \varepsilon \leq x \leq m$. Then $[a,x]$ is covered by a finite subfamily $\{U_k\}_{k\in K}$, and with $[x, m+\varepsilon] \subseteq U_j$, we get $[a, m+\varepsilon]$ covered by the finite subfamily $\{U_k\}_{k\in K} \cup \{U_j\}$. One sees that $m+\varepsilon \in [a,b]$ contradicts the fact that m is the supremum in $[a,b]$. Hence $m = b$, and $[a,b]$ is covered by a finite subfamily of $\{U_i\}_{i\in I}$. We deduce the compactness of $[a,b]$ from Proposition 4.3. □

4.2 Properties of Compact Spaces

Proposition 4.5. *Let X be a set equipped with a filter base $\mathscr{B}$, Y a compact space, and $f : X \to Y$ a function. Then f admits at least one adherence value along $\mathscr{B}$.*

Proof. Consider the family $\{\overline{f(B)}\}_{B\in\mathscr{B}}$ of closed subsets of Y, and let $A = \bigcap_{B\in\mathscr{B}} \overline{f(B)}$. If $A = \varnothing$, there exist $B_1, \dots, B_n \in \mathscr{B}$ such that $\overline{f(B_1)} \cap \dots \cap \overline{f(B_n)} = \varnothing$ as Y is compact. Now, there exists $B \in \mathscr{B}$ such that $B \subseteq B_1 \cap \dots \cap B_n$, whence $f(B) \subseteq f(B_1) \cap \dots \cap f(B_n)$, and consequently $f(B_1) \cap \dots \cap f(B_n) \neq \varnothing$. This contradiction proves that $A \neq \varnothing$, so we get the result by using Proposition 2.9. □

Proposition 4.6. *Let X be a set equipped with a filter base $\mathscr{B}$, Y a compact space, $f : X \to Y$ a function, and A the set of adherence values of f along $\mathscr{B}$. Take an open subset U of Y containing A. Then, there exists $B \in \mathscr{B}$ such that $f(B) \subseteq U$.*

Proof. One has $(Y \setminus U) \cap A = \varnothing$, meaning that $(Y \setminus U) \cap \bigcap_{B\in\mathscr{B}} \overline{f(B)} = \varnothing$. Since Y is compact, there exist $B_1, \dots, B_n \in \mathscr{B}$ such that $(Y \setminus U) \cap \overline{f(B_1)} \cap \dots \cap \overline{f(B_n)} = \varnothing$. Furthermore, there exist $B \in \mathscr{B}$ such that $B \subseteq B_1 \cap \dots \cap B_n$. Then $(Y \setminus U) \cap \overline{f(B)} = \varnothing$, implying $\overline{f(B)} \subseteq U$. □

Corollary 4.7. *Let X be a set equipped with a filter base $\mathscr{B}$, Y a compact space, and $f : X \to Y$ a function. If f admits only one adherence value l along $\mathscr{B}$, then f tends to l along $\mathscr{B}$.*

Proof. From Proposition 4.6, for any neighborhood V of l, there exists $B \in \mathscr{B}$ such that $f(B) \subseteq V$. □

Proposition 4.8. *Let X be a compact space, and A a closed subspace of X. Then A is compact.*

Proof. Let $\{A_i\}_{i\in I}$ be a family of closed subsets of A with empty intersection. We know from Proposition 3.3 that each A_i is the intersection of A with a closed subset of X then is closed in X. Since X is compact, there exists a finite subfamily $\{A_j\}_{j\in J}$ with empty intersection. □

Proposition 4.9. *Let X be a separated space, and A a compact subspace of X. Then A is closed in X.*

Proof. Take $x \in X \setminus A$. For every $y \in A$, there exist open neighborhoods U_y, V_y of x, y respectively in X that are disjoint. We have $A \subseteq \bigcup_{y\in A} V_y$, and since A is compact, there exist $y_1, \dots, y_n \in A$ such that $A \subseteq V_{y_1} \cup \dots \cup V_{y_n}$. The set $U_{y_1} \cap \dots \cap U_{y_n}$ is an open neighborhood of x contained in $X \setminus A$. It follows that $X \setminus A$ is neighborhood of each of its points, and is consequently open from Proposition 1.9. Therefore A is closed in X. □

Proposition 4.10. *Let X be a separated space.*

(i) If A, B are compact subsets of X, then $A \cup B$ is compact.

(ii) If $\{A_i\}_{i\in I}$ is a nonempty family of compact subsets of X, then $\bigcap_{i\in I} A_i$ is compact.

Proof. (i) : Let $\{U_i\}_{i\in I}$ be a covering of $A\cup B$ by open subsets of X. There exist finite subsets J_1, J_2 of I such that $\{U_j\}_{j\in J_1}$ covers A and $\{U_j\}_{j\in J_2}$ covers B. Then $\{U_j\}_{j\in J_1\cup J_2}$ covers $A\cup B$, and we deduce from Proposition 4.3 that $A\cup B$ is compact.

(ii) : We know from Proposition 4.9 that each A_i is closed in X. Therefore $\bigcap_{i\in I} A_i$ is closed in X, and consequently in each A_i. Since each A_i is compact, then $\bigcap_{i\in I} A_i$ is compact by Proposition 4.8. □

Proposition 4.11. *Let X be a separated compact space. Every point of X has a fundamental system of compact neighborhoods.*

Proof. Take a point x_0 and an open neighborhood A of x_0 in X. The sets $\{x_0\}$ and $X\setminus A$ are disjoint compact subsets of X. For every $x\in X\setminus A$, there exist disjoint open subsets U_x, V_x of X such that $x_0\in U_x$ and $x\in V_x$. Since $X\setminus A\subseteq \bigcup_{x\in X\setminus A} V_x$, there exists $x_1,\dots,x_n\in X\setminus A$ such that $X\setminus A\subseteq V_{x_1}\cup\dots\cup V_{x_n}$. Then, $U=U_{x_1}\cap\dots\cap U_{x_n}$ and $V=V_{x_1}\cup\dots\cup V_{x_n}$ are disjoint open subsets of X such that $x_0\in U$ and $X\setminus A\subseteq V$. Hence $\overline{U}$ is a compact neighborhood of x_0. We have $U\subseteq X\setminus V$, therefore $\overline{U}\subseteq X\setminus V$ as $X\setminus V$ is closed, and consequently $\overline{U}\subseteq A$. □

Proposition 4.12. *Let X be a compact space, Y a topological space, and $f: X\to Y$ a continuous function. Then $f(X)$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be a family of open subsets of Y covering $f(X)$. Since f is continuous, then each $f^{-1}(U_i)$ is an open subset of X from Proposition 2.14. Moreover, $X=\bigcup_{i\in I} f^{-1}(U_i)$, then there exists a finite subset J of I such that $X=\bigcup_{j\in J} f^{-1}(U_j)$. Hence $\{U_j\}_{j\in J}$ covers $f(X)$, and $f(X)$ is therefore compact. □

Corollary 4.13. *Let X be a compact space, Y a separated space, and $f: X\to Y$ a continuous bijective function. Then f is a homeomorphism of X onto Y.*

Proof. If A is a closed subset of X, then A is compact from Proposition 4.8, therefore $f(A)$ is compact from Proposition 4.12, and consequently closed from Proposition 4.9. We deduce from Proposition 2.14 that f^{-1} is continuous. □

Theorem 4.14. *The product of a finite number of compact spaces is compact.*

Proof. It suffices to show that if X and Y are compact, then $X\times Y$ is compact. Let $\{U_i\}_{i\in I}$ be a covering of $X\times Y$ with open subsets. For every $m=(x,y)\in X\times Y$, fix an open set U_m such that $m\in U_m$. By Proposition 3.11, there exist an open neighborhood V_m of x in X and an open neighborhood W_m of y in Y such that $V_m\times W_m\subseteq U_m$.

For a fixed $x_0\in X$, $\{x_0\}\times Y$ is homeomorphic to Y. Indeed, the function $y\mapsto (x_0,y)$ of Y onto $\{x_0\}\times Y$ is bijective. It is continuous from Y into $X\times Y$ by Proposition 3.14 , therefore from Y into $\{x_0\}\times Y$ by Corollary 3.8. Its inverse function is the composite of the canonical injection of $\{x_0\}\times Y$ into $X\times Y$, which is continuous from Corollary 3.8 once again, and of the canonical projection of $X\times Y$ onto Y, which is also continuous from Proposition 3.14. The set $\{x_0\}\times Y$ is then compact.

The family of open subsets $\{V_m\times W_m\}_{m\in\{x_0\}\times Y}$ is a covering of $\{x_0\}\times Y$, so there consequently exist finite points $m_1,\dots,m_n\in\{x_0\}\times Y$ such that $\{x_0\}\times Y\subseteq (V_{m_1}\times W_{m_1})\cup\dots\cup(V_{m_n}\times W_{m_n})$. The intersection $A_{x_0}=V_{m_1}\cap\dots\cap V_{m_n}$ is an open neighborhood of x_0. For every $(x,y)\in A_{x_0}\times Y$, there exists

$k \in \{1,\dots,n\}$ such that $(x,y) \in V_{m_k} \times W_{m_k}$, hence $A_{x_0} \times Y$ is covered by a finite subset of $\{U_i\}_{i\in I}$. Now $\{A_{x_0}\}_{x_0\in X}$ forms a covering of X, from which one can extract a finite covering of open subsets $\{A_{x_1},\dots,A_{x_p}\}$. Each $A_{x_j} \times Y$, with $j \in \{1,\dots,p\}$, is covered by a finite subset of $\{U_i\}_{i\in I}$, therefore $X \times Y$ is covered by a finite subset of $\{U_i\}_{i\in I}$. □

4.3 Locally Compact Spaces

Definition 4.15. A topological space X is said to be **locally compact** if every point of X admits a compact neighborhood.

Example. Consider the product topological space $\mathbb{R}^n$, where $\mathbb{R}$ is equipped with the usual topology, and take $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. We know from the theorem of Borel-Lebesgue that, for every $i \in \{1,\dots,n\}$, $[x_i - 1, x_i + 1]$ is a compact neighborhood of x_i in $\mathbb{R}$. Then, by Proposition 3.11 and Theorem 4.14, $[x_1 - 1, x_1 + 1] \times \dots \times [x_n - 1, x_n + 1]$ is a compact neighborhood of x. The topological space $\mathbb{R}^n$ is therefore locally compact.

Proposition 4.16. *Let X be a separated space. The following conditions are equivalent:*

- (i) *X is locally compact,*
- (ii) *every point of X admits a fundamental system of compact neighborhoods.*

Proof. We obviously have $(ii) \Rightarrow (i)$. We only prove $(i) \Rightarrow (ii)$: Let $x \in X$ and V be a compact neighborhood of x. We know from Proposition 4.11 that x admits in V a fundamental system $\{V_i\}_{i\in I}$ of compact neighborhoods. We deduce from Proposition 3.4 that $\{V_i\}_{i\in I}$ is a fundamental system of compact neighborhoods of x in X. □

Proposition 4.17. *Let X be a locally compact space, and Y a subspace of X.*

- (i) *If Y is closed, then Y is locally compact.*
- (ii) *If X is separated and Y is open, then Y is locally compact.*

Proof. Let $x \in Y$ and V a compact neighborhood of x in X. Then $V \cap Y$ is a neighborhood of x in Y.

(i) : We know from Proposition 3.3 that $V \cap Y$ is closed in V, hence is compact by Proposition 4.8.

(ii) : As Y is a neighborhood of x, we can suppose from Proposition 4.16 that $V \subseteq Y$, and then V is a compact neighborhood of x in Y. □

Proposition 4.18. *Let $X_1,\dots,X_n$ be locally compact spaces, and $X = X_1 \times \dots \times X_n$. Then X is locally compact.*

Proof. Take $x = (x_1,\dots,x_n) \in X$. For every $i \in \{1,\dots,n\}$, there exists a compact neighborhood V_i of x_i in X_i. Then $V_1 \times \dots \times V_n$ is a neighborhood of x in X is compact by Theorem 4.14. □

Chapter 5

Connected Spaces

5.1 Connected Spaces

Definition 5.1. A topological space X is said to be **connected** if there does not exist a pair (A,B) of disjoint nonempty open subsets of X such that $X = A \sqcup B$. By passage to complements, this definition is equivalent to the nonexistence of a pair (A,B) of disjoint nonempty closed subsets of X such that $X = A \sqcup B$. It is also equivalent to the nonexistence of a subset of X, distinct from X and $\varnothing$, that is both open and closed.

Proposition 5.2. *The topological space $\mathbb{R}$ equipped with the usual topology is connected.*

Proof. Let A be an open and closed subset of $\mathbb{R}$, and assume A and $\mathbb{R} \setminus A$ nonempty. Taking $x \in \mathbb{R} \setminus A$, one of the sets $A \cap [x, +\infty)$ and $A \cap (-\infty, x]$ is nonempty. Suppose that $B = A \cap [x, +\infty) \neq \varnothing$. Then B is closed. Since it is bounded below, then it has a smallest element as its infimum b is adherent to B. Besides, since $B = A \cap (x, +\infty)$, then B is also open. Hence B contains an interval $(b - \varepsilon, b + \varepsilon)$ with $\varepsilon > 0$. That contradicts the fact that b is the smallest element of B. □

Definition 5.3. Let X be a topological space and $Y \subseteq X$. One says that Y is a **connected subset** of X if the topological space Y is connected.

Example. The subspace $\mathbb{Q}$ of $\mathbb{R}$ is not connected. Take indeed an element $x \in \mathbb{R} \setminus \mathbb{Q}$ such as $\sqrt{2}$ or π. Then $\mathbb{Q} = \big((-\infty, x) \cap \mathbb{Q}\big) \sqcup \big((x, +\infty) \cap \mathbb{Q}\big)$ which are two disjoint open subsets of $\mathbb{Q}$.

Proposition 5.4. *Let X be a topological space, $\{A_i\}_{i \in I}$ a family of connected subsets of X, and A the set $\bigcup\limits_{i \in I} A_i$. If the A_i intersect pairwise, then A is connected.*

Proof. Suppose A is not connected. There exist nonempty subsets $U, V \subseteq A$ open in A such that $V = A \setminus U$. For every $i \in I$, $U \cap A_i$ and $V \cap A_i$ are both open and complementary in A_i. Since A_i is connected, then $U \cap A_i = \varnothing$ or $V \cap A_i = \varnothing$. Let I_U and I_V be the set of $i \in I$ such that $A_i \subseteq U$ and $A_i \subseteq V$ respectively. Then, $U = \bigcup\limits_{i \in I_U} A_i$ and $V = \bigcup\limits_{i \in I_V} A_i$. Therefore, there exist $i, j \in I$, $i \neq j$, such that A_i and A_j are disjoint, which is a contradiction. □

Corollary 5.5. *Let X be a topological space, and $A_1, \ldots, A_n$ connected subspaces of X such that $A_i \cap A_{i+1} \neq \varnothing$ if $i \in \{1, \ldots, n\}$. Then, $A_1 \cup \cdots \cup A_n$ is connected.*

Proof. The proof is by induction. We suppose that $A_1 \cup \cdots \cup A_{n-1}$ is connected. As $A_{n-1} \cap A_n \neq \varnothing$, we deduce from Proposition 5.4 that $A_1 \cup \cdots \cup A_n$ is connected. □

Proposition 5.6. *Let X be a topological space, A a connected subset of X, and B a subset of X such that $A \subseteq B \subseteq \overline{A}$. Then B is connected.*

Proof. Suppose that B is the union of subsets U, V that are disjoint and open in B. There exist open sets U', V' in X such that $U = B \cap U'$ and $V = B \cap V'$. The sets $A \cap U$ and $A \cap V$ are then open and complementary in A. Since A is connected, we have for example $A \cap U = \varnothing$, then $A \cap U' = \varnothing$, in other words $A \subseteq X \setminus U'$. Since $X \setminus U'$ is closed, then $\overline{A} \subseteq X \setminus U'$. So $B \cap U' = \varnothing$, implying $U = \varnothing$. □

Proposition 5.7. *Let X, Y be topological spaces and f a continuous function from X into Y. If X is connected, then $f(X)$ is connected.*

Proof. If $f(X)$ is not connected, it has nonempty open subsets $U, V \subseteq f(X)$ that are complementary. So $f^{-1}(U), f^{-1}(V) \subseteq X$ are nonempty open subsets that are complementary, which is absurd. □

Proposition 5.8. *Consider $\mathbb{R}$ equipped with the usual topology, and $A \subseteq R$. The following conditions are equivalent:*

(i) *A is connected,*

(ii) *A is an interval.*

Proof. We can assume that A is nonempty and not reduced to a point.

$(ii) \Rightarrow (i)$: If A is open, then A is homeomorphic to $\mathbb{R}$, and consequently connected by Proposition 5.2. If A is an arbitrary interval, then $A^\circ \subseteq A \subseteq \overline{A}$, and consequently connected by Proposition 5.6.

$(i) \Rightarrow (ii)$: Suppose that A is not an interval. There exist $a, b \in A$ and $x_0 \in \mathbb{R} \setminus A$ such that $a < x_0 < b$. Then A is the union of the sets $A \cap (-\infty, x_0)$ and $A \cap (x_0, +\infty)$ which are open in A. Since A is connected, $A \cap (x_0, +\infty)$ for example is empty. Then $x < x_0$ for all $x \in A$, which contradicts $b \in A$. □

Proposition 5.9. *Let X be a connected topological space, $f : X \to \mathbb{R}$ a continuous function, and $a, b \in X$. Then f takes on every value between $f(a)$ and $f(b)$.*

Proof. The set $f(X)$ is a connected subset of $\mathbb{R}$ by Proposition 5.7, hence is an interval of $\mathbb{R}$ by Proposition 5.8. This interval contains $f(a)$ and $f(b)$, hence all numbers between them. □

5.2 Connected Components

Proposition 5.10. *Let X be a topological space, and $x \in X$. Among the connected subspaces of X containing x, there exists one that is larger than all the others.*

Proof. The union of all the connected subsets of X containing x is connected by Proposition 5.4, and is obviously the largest of the connected subsets of X containing x. □

Definition 5.11. Let X be a topological space and $x \in X$. The largest connected subset of X containing x is called the **connected component** of x in X.

Example. The topological spaces $X = \mathbb{R} \setminus \{0\}$ and $Y = \mathbb{R} \setminus \{0, 1\}$ are not homeomorphic, since X has the two connected components $(-\infty, 0)$, $(0, +\infty)$, while Y the has three $(-\infty, 0)$, $(0, 1)$, $(1, +\infty)$.

Proposition 5.12. *Let X be a topological space.*

(i) *Every connected component of X is closed in X.*

(*ii*) *Two distinct connected components are disjoint.*

Proof. (*i*) : If A_x is the connected component of x, then $\overline{A_x}$ is connected by Proposition 5.6. But A_x is the largest connected subset of X containing x, hence $\overline{A_x} = A_x$.

(*ii*) : Let A_x, A_y be connected components that are not disjoint. Then $A_x \cup A_y$ is connected by Proposition 5.4. Since $x \in A_x \cup A_y$, then $A_x \cup A_y \subseteq A_x$, hence $A_y \subseteq A_x$. Similarly $A_x \subseteq A_y$, therefore $A_x = A_y$. □

Proposition 5.13. *Let X be a topological space. If every point of X has a connected neighborhood, the connected components of X are open.*

Proof. Let C be a connected component of X, $x \in C$, and V a connected neighborhood of x. Since $x \in C \cap V$, the union $C \cup V$ is then connected, and $C \cup V \subseteq C$. Hence $V \subseteq C$, and C is a neighborhood of x. We deduce from Proposition 1.9 that C is open. □

5.3 Locally Connected Spaces

Definition 5.14. A topological space X is said to be **locally connected** at its point x if x has a fundamental system of connected neighborhoods. If X is locally connected at each of its points, it is said to be locally connected.

Example. The topological space $\mathbb{R} \setminus \{0\}$ is not connected, but it is locally connected.

Proposition 5.15. *Let X be a topological space. The following conditions are equivalent:*

(*i*) *X is locally connected,*

(*ii*) *for every open set V of X, each connected component of V is open in X.*

Proof. (*i*) ⇒ (*ii*) : Let C be a connected component of an open set V in X, and $x \in C$. We can choose a connected neighborhood U of x such that $U \subseteq V$. Since U is connected, it must lie entirely in C. We deduce from Proposition 1.9 that C is open.

(*ii*) ⇒ (*i*) : Given $x \in X$, a neighborhood V of x in X, and open set U such that $x \in U$ and $U \subseteq V$. Let C be the connected component of U containing x. Since C is connected and open in X, then it is a connected neighborhood of x contained in V. □

5.4 Path Connected Spaces

Definition 5.16. Let X be a topological space and $a, b \in X$. A continuous map f from $[0,1]$ into X such that $f(0) = a$ and $f(1) = b$ is called a **path** in X with **origin** a and **extremity** b. If any two points of X are the origin and extremity of a path in X, X is said to be **path connected**.

Example. The open unit n-ball $\mathbb{B}^n := \left\{(x_1, \dots, x_n) \in \mathbb{R}^n \,\middle|\, x_1^2 + \cdots + x_n^2 < 1\right\}$ is path connected. Indeed, any points $x, y \in \mathbb{B}^n$ can be connected by the straight-line path $f : [0,\, 1] \to \mathbb{B}^n$ defined by

$$f(t) = (1-t)x + ty.$$

Proposition 5.17. *Let X be an path connected topological space. Then X is connected.*

Proof. Take a point $x_0 \in X$. For every $x \in X$, let $f_x : [0,1] \to X$ be a path with origin x_0 and extremity x. Since $[0,1]$ is connected by Proposition 5.8, then $f_x\big([0,1]\big)$ is connected by Proposition 5.7. Therefore $X = \bigcup_{x \in X} f_x\big([0,1]\big)$ is connected by Proposition 5.4, as x_0 belongs to all of the $f_x\big([0,1]\big)$. □

Proposition 5.18. *Let X be a topological space, and $A,B \subseteq X$. If A,B are path connected such that $A \cap B \neq \varnothing$, then $A \cup B$ is path connected.*

Proof. Let $x \in A$, $y \in B$, and pick $z \in A \cap B$. Choose paths $f : [0,1] \to A$, $g : [0,1] \to B$ such that $f(0) = x$, $f(1) = z$, and $g(0) = z$, $g(1) = y$. We obtain a path $h : [0,1] \to A \cup B$ from x to y as follows:

$$h(t) = \begin{cases} f(2t) & \text{if } t \in \big[0, \frac{1}{2}\big], \\ g(2t-1) & \text{if } t \in \big[\frac{1}{2}, 1\big]. \end{cases}$$

□

Proposition 5.19. *Let X,Y be topological spaces, and $f : X \to Y$ a continuous function. If X is path connected, then $f(X)$ is path connected.*

Proof. If $y_1, y_2 \in f(X)$, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. As X is path connected, there exists a path $h : [0,\, 1] \to X$ from x_1 to x_2. Hence $f \circ h : [0,\, 1] \to Y$ is a path from y_1 to y_2. □

5.5 Locally Path-Connected Spaces

Definition 5.20. A topological space X is said to be **locally path connected** at its point x if x has a fundamental system of path-connected neighborhoods. If X is locally path connected at each of its points, it is said to be locally path connected.

Definition 5.21. Let X be a topological space and $x \in X$. The **path component** of x in X is the set formed by the points $y \in X$ such that a path with origin x and extremity y in X exists.

Proposition 5.22. *Let X be a topological space. The following conditions are equivalent:*

- (i) *X is locally path connected,*
- (ii) *for every open set V of X, each path component of V is open in X.*

Proof. $(i) \Rightarrow (ii)$: Let C be a path component of an open set V in X, and $x \in C$. We can choose a path-connected neighborhood U of x such that $U \subseteq V$. Since U is path connected, it must lie entirely in C. We deduce from Proposition 1.9 that C is open.

$(ii) \Rightarrow (i)$: Given $x \in X$, a neighborhood V of x in X, and open set U such that $x \in U$ and $U \subseteq V$. Let C be the path component of U containing x. Since C is path connected and open in X, then it is a path-connected neighborhood of x contained in V. □

Chapter 6

Metric Spaces

6.1 Metric Spaces

Definition 6.1. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}_+$ satisfying the following conditions:

(i) $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$,

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

A set equipped with a metric is called a **metric space**.

Example. Let $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, $y = (x_1,\ldots,x_n) \in \mathbb{R}^n$, and set $d(x,y) = \sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$. It is known that d is a metric on $\mathbb{R}^n$, and in this way $\mathbb{R}^n$ becomes a metric space.

Definition 6.2. Let X be a set equipped with a metric d, and $Y \subseteq X$. Then Y becomes a metric space with the restriction of d to $Y \times Y$, and is called a **metric subspace** of X.

Definition 6.3. Let X be a metric space with metric d, take $a \in X$, and $\rho \in \mathbb{R}_+^*$. The set $B(a,\rho) := \{x \in X \mid d(a,x) < \rho\}$ is called an **open ball** with center a and radius ρ. A subset $A \subseteq X$ is said to be **open** if, for each $x_0 \in A$, there exists $\varepsilon \in \mathbb{R}_+^*$ such that $B(x_0, \varepsilon) \subseteq A$.

Definition 6.4. Let X be a metric space, and $A \subseteq X$. One says that A is **closed** if $X \setminus A$ is open.

Proposition 6.5. *Every metric space X is a topological space, and the topology of X is formed by the open sets of X.*

Proof. Let X be a metric space. The subsets $\varnothing$ and X of X are clearly open.
Take a family $\{A_i\}_{i\in I}$ of open subsets of X. Let $A = \bigcup_{i\in I} A_i$, and $x_0 \in A$. There exists $i \in I$ such that $x_0 \in A_i$. Hence, there exists $\varepsilon \in \mathbb{R}_+^*$ such that $B(x_0, \varepsilon) \subseteq A_i \subseteq A$. Thus A is open.
Suppose now that I is finite. Let $C = \bigcap_{i\in I} A_i$, and $x_0 \in C$. For every $i \in I$, there exists $\varepsilon_i \in \mathbb{R}_+^*$ such that $B(x_0, \varepsilon_i) \subseteq A_i$. If $\varepsilon \in \inf\{\varepsilon_i\}_{i\in I}$, then $B(x_0, \varepsilon) \subseteq A_i$ for every $i \in I$. Hence $B(x_0, \varepsilon) \subseteq C$, and C is consequently open. □

Proposition 6.6. *Let X be a set, and d, d' metrics on X. Suppose there exist $c, c' \in \mathbb{R}_+^*$ such that*

$$c\,d(x,y) \leq d'(x,y) \leq c'\,d(x,y)$$

for all $x, y \in X$. The open subsets of X are the same for d and d'.

Proof. Let A be a subset of X that is open for d, and $x_0 \in A$. There exists $\varepsilon \in \mathbb{R}_+^*$ such that $\{x \in X \mid d(x_0, x) < \varepsilon\} \subseteq A$. If $x \in X$ satisfies $d'(x_0, x) < c\varepsilon$, then $d(x_0, x) < \varepsilon$, so $x \in A$. Hence A is also open for d'. On the other side, one proves that if A is open for d', then A is open for d by interchanging the roles of d and d'. □

6.2 Continuity of the Metric

Proposition 6.7. *Let X be a metric space. Its metric $d : X \times X \to \mathbb{R}_+$ is continuous.*

Proof. Let $(x_0, y_0) \in X \times X$, and take $\varepsilon \in \mathbb{R}_+^*$. The set $B\big(x_0, \frac{\varepsilon}{2}\big) \times B\big(y_0, \frac{\varepsilon}{2}\big)$ is a neighborhood of (x_0, y_0) in $X \times X$. If $(x, y) \in B\big(x_0, \frac{\varepsilon}{2}\big) \times B\big(y_0, \frac{\varepsilon}{2}\big)$, then

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) < \frac{\varepsilon}{2} + d(x_0, y_0) + \frac{\varepsilon}{2} = d(x_0, y_0) + \varepsilon,$$

$$d(x_0, y_0) \leq d(x_0, x) + d(x, y) + d(y, y_0) < \frac{\varepsilon}{2} + d(x, y) + \frac{\varepsilon}{2} = d(x, y) + \varepsilon,$$

therefore $|d(x, y) - d(x_0, y_0)| < \varepsilon$. So d is continuous at (x_0, y_0). □

Definition 6.8. Let X be a metric space, and A a nonempty subset of X. One calls **diameter** of A the number $\mathrm{diam}(A) := \sup\{d(x, y) \mid x, y \in A\}$.

Lemma 6.9. *Consider $\mathbb{R}$ with the usual topology, and let A be a nonempty subset of $\mathbb{R}$. Suppose that A is bounded above, and x its supremum. Then x is the largest element of $\overline{A}$.*

Proof. Let V be a neighborhood of x in $\mathbb{R}$, and $\varepsilon \in \mathbb{R}_+^*$ such that $(x-\varepsilon, x+\varepsilon) \subseteq V$. By definition of the supremum, there exists $y \in A$ such that $x - \varepsilon < y \leq x$. Then $y \in V$, meaning that $V \cap A \neq \varnothing$, thus x is adherent to A.
Let $x' \in \overline{A}$ such that $x' > x$, and set $\varepsilon = x' - x > 0$. Then $(x' - \varepsilon, x' + \varepsilon)$ is a neighborhood of x', therefore intersects A. Let $y \in (x' - \varepsilon, x' + \varepsilon) \cap A$. Since $y > x' - \varepsilon = x$, x is then not an upper bound for A, which is absurd. So, x is the largest element of $\overline{A}$. □

Proposition 6.10. *Let X be a metric space, and $A \subseteq X$. The sets A and $\overline{A}$ have the same diameter.*

Proof. Denote d the metric of X. Let $D = \{d(x, y) \mid x, y \in A\}$ and $D' = \{d(x, y) \mid x, y \in \overline{A}\}$. We obviously have $D \subseteq D'$. One deduce from Proposition 3.11 that every point of $\overline{A} \times \overline{A}$ is adherent to $A \times A$. So $D' = d(\overline{A} \times \overline{A}) \subseteq d(\overline{A \times A})$, and $d(\overline{A \times A}) \subseteq \overline{d(A \times A)} = \overline{D}$ by Proposition 2.14 and Proposition 6.7. Then $D' \subseteq \overline{D}$, and consequently $\overline{D} = \overline{D}'$. If D is bounded, we then deduce from Lemma 6.9 that the diameter of A and $\overline{A}$ is the largest element of $\overline{D}$. If D is unbounded, then D and D' have the same supremum $+\infty$. □

Definition 6.11. Let X be a metric space with metric d, and A, B two nonempty subsets of X. The **distance** from A to B the number $d(A, B) := \inf\{d(x, y) \mid x \in A, y \in B\}$. It is clear that $d(A, B)$ and $d(B, A)$ are equal. If $z \in X$, we define $d(z, A) := \inf\{d(z, x) \mid x \in A\}$.

6.3 Sequences in Metric Spaces

Proposition 6.12. *Let X be a metric space, $x \in X$, and $A \subseteq X$. The following conditions are equivalent:*

(i) $x \in \overline{A}$,

(ii) *there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points in A that tends to x.*

Proof. $(ii) \Rightarrow (i)$: Since every neighborhood of x intersects $\{x_n\}_{n\in\mathbb{N}}$, then every neighborhood of x intersects A which means that $x \in \overline{A}$.

$(i) \Rightarrow (ii)$: For every $n \in \mathbb{N}$, there exists a point $x_n \in A \cap B\big(x, \frac{1}{n}\big)$. Then $(x_n)_{n\in\mathbb{N}}$ tends to x. □

Proposition 6.13. *Let X be a metric space, $(x_n)_{n\in\mathbb{N}}$ a sequence of points in X, and $x \in X$. The following conditions are equivalent:*

(i) *x is an adherence value of $(x_n)_{n\in\mathbb{N}}$ along the filter base $\big\{\{n, n+1, \dots\}\big\}_{n\in\mathbb{N}}$,*

(ii) *there exists an infinite subset $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\mathbb{N}$, with $n_k < n_{k+1}$, such that $(x_{n_k})_{k\in\mathbb{N}}$ tends to x along the filter base $\big\{\{n_k, n_{k+1}, \dots\}\big\}_{k\in\mathbb{N}}$.*

Proof. $(ii) \Rightarrow (i)$: The point x is then an adherence value of $(x_{n_k})_{k\in\mathbb{N}}$, and consequently of $(x_n)_{n\in\mathbb{N}}$.

$(i) \Rightarrow (ii)$: If d is the metric of X, there exist $n_1 \in \mathbb{N}$ such that $d(x_{n_1}, x) < 1$, $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and $d(x_{n_2}, x) < \frac{1}{2}$, $n_3 \in \mathbb{N}$ such that $n_3 > n_2$ and $d(x_{n_3}, x) < \frac{1}{3}$, and so on. So, the sequence $(x_{n_k})_{k\in\mathbb{N}}$ tends to x along $\big\{\{n_k, n_{k+1}, \dots\}\big\}_{k\in\mathbb{N}}$. □

Proposition 6.14. *Let X,Y be metric spaces, $A \subseteq X$, $f : A \to Y$ a function, $a \in \overline{A}$, and $y \in Y$. The following conditions are equivalent:*

(i) *the point y is an adherence value of f along the filter $\{A \cap V\}_{V\in\mathscr{V}}$, where $\mathscr{V}$ is a fundamental system of neighborhoods of a,*

(ii) *there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in A such that $(x_n)_{n\in\mathbb{N}}$ tends to a and $\big(f(x_n)\big)_{n\in\mathbb{N}}$ tends to y.*

Proof. $(ii) \Rightarrow (i)$: On one side, if $V \in \mathscr{V}$, there exists $i \in \mathbb{N}$ such that $x_n \in A \cap V$ if $n \geq i$. On the other side, if W is a neighborhood of y, there exists $j \in \mathbb{N}$ such that $f(x_n) \in W$ if $n \geq j$. Then, $f(x_n) \in f(A \cap V) \cap W$ if $n \geq \max\{i, j\}$.

$(i) \Rightarrow (ii)$: Denote by $B_X(a, \rho)$ and $B_Y(y, \rho')$ the open balls of centers and radius a,y and ρ,ρ' respectively. Take a point $x_1 \in B_X(a, 1) \cap A$ such that $f(x_1) \in B_Y(y, 1)$, take a point $x_2 \in B_X(a, \frac{1}{2}) \cap A$ such that $f(x_2) \in B_Y(y, \frac{1}{2})$, take a point $x_3 \in B_X(a, \frac{1}{3}) \cap A$ such that $f(x_3) \in B_Y(y, \frac{1}{3})$, and so on. Hence, the sequence $(x_n)_{n\in\mathbb{N}}$ tends to a, and $\big(f(x_n)\big)_{n\in\mathbb{N}}$ tends to y. □

Proposition 6.15. *Let X,Y be metric spaces, $f : X \to Y$ a function, and $x \in X$. The following conditions are equivalent:*

(i) *f is continuous at x,*

(ii) *for every sequence $(x_n)_{n\in\mathbb{N}}$ in X that tends to x, the sequence $\big(f(x_n)\big)_{n\in\mathbb{N}}$ tends to $f(x)$.*

Proof. $(i) \Rightarrow (ii)$: Consider the filter base $\big\{\{x_n, x_{n+1}, \dots\}\big\}_{n\in\mathbb{N}}$ and a neighborhood V of $f(x)$ in Y. There exists a neighborhood U of x in X such that $f(U) \subseteq V$. And there exists $k \in \mathbb{N}$ such that $\{x_k, x_{k+1}, \dots\} \subseteq U$. Then, $\big\{f(x_k), f(x_{k+1}), \dots\big\} \subseteq V$.

$(ii) \Rightarrow (i)$: Let d_X and d_Y be the metrics of X and Y respectively, and suppose that f is not continuous at x. There exists $\varepsilon \in \mathbb{R}_+^*$ such that, for any $\eta \in \mathbb{R}_+^*$, there is $y \in X$ with $d_X(x, y) < \eta$ yet $d_Y\big(f(x), f(y)\big) > \varepsilon$. If we successively take $\eta = 1, \frac{1}{2}, \frac{1}{3}, \dots$, we obtain points $y_1, y_2, y_3, \dots$ of X such that $d_X(x, y_n) < \frac{1}{n}$ and $d_Y\big(f(x), f(y_n)\big) > \varepsilon$ for $n \in \mathbb{N}$. Then $(y_n)_{n\in\mathbb{N}}$ tends to x, but $\big(f(y_n)\big)_{n\in\mathbb{N}}$ does not tend to $f(x)$. □

6.4 Complete Metric Spaces

Definition 6.16. Let X be a metric space with metric d. A sequence $(x_n)_{n\in\mathbb{N}}$ of points in X is called a **Cauchy sequence** if, for every $\varepsilon \in \mathbb{R}_+^*$, there exists $p \in \mathbb{N}$ such that $m,n \geq p$ implies $d(x_m, x_n) < \varepsilon$.

Proposition 6.17. *Let X be a metric space with metric d. If a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X has a limit in X, then it is a Cauchy sequence.*

Proof. Suppose that $(x_n)_{n\in\mathbb{N}}$ tends to x. For every $\varepsilon \in \mathbb{R}_+^*$, there exists a positive integer p such that $n \geq p$ implies $d(x_n, x) < \frac{\varepsilon}{2}$. Then, if m,n are positive integers bigger than p, we have $d(x_m, x) < \frac{\varepsilon}{2}$ and $d(x_n, x) < \frac{\varepsilon}{2}$, which implies $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \varepsilon$. □

Definition 6.18. A metric space X is said to be **complete** if every Cauchy sequence of points in X has a limit in X.

Proposition 6.19. *Let X be a metric space, $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence in X, and $(x_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(x_n)_{n\in\mathbb{N}}$. If the sequence $(x_{n_k})_{k\in\mathbb{N}}$ has a limit l, then $(x_n)_{n\in\mathbb{N}}$ also tends to l.*

Proof. For every $\varepsilon \in \mathbb{R}_+^*$, there exists a positive integer p such that, if m,n are positive integers bigger than p, then $d(x_m, x_n) < \frac{\varepsilon}{2}$. Fix a positive integer n bigger than p. Since $(x_{n_k})_{k\in\mathbb{N}}$ tends to l, then $\big(d(x_{n_k}, x_n)\big)_{k\in\mathbb{N}}$ tends to $d(l, x_n)$, so $d(l, x_n) \leq \frac{\varepsilon}{2} < \varepsilon$. As this is true for all positive integers $n \geq p$, then $(x_n)_{n\in\mathbb{N}}$ also tends to l. □

Proposition 6.20. *Let X be a complete metric space, and Y a closed subspace of X. Then Y is complete.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in Y. It is also a Cauchy sequence in X, hence has a limit l in X. We deduce from Proposition 6.12 that $l \in \overline{Y}$. But $\overline{Y} = Y$, thus $(x_n)_{n\in\mathbb{N}}$ has a limit in Y. □

Proposition 6.21. *Let X be a metric space, and Y a complete metric subspace of X. Then Y is closed in X.*

Proof. Take $l \in \overline{Y}$. We know from Proposition 6.12 that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in Y that tends to l. So, we deduce Proposition 6.17 that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. It thus has a limit in Y since Y is complete. As l is its limit, we must have $l \in Y$, therefore $\overline{Y} = Y$. □

Part II

Algebraic Topology

Chapter 7

Fundamental Groups

7.1 Homotopy of Paths

Definition 7.1. Let X be a topological space, and f, g two paths in X. These paths are said to be **path homotopic** if they have the same origin a, the same extremity b, and if there is a continuous function $F : [0,1] \times [0,1] \to X$ such that, if $s, t \in [0,1]$,

$$\begin{aligned} F(s,0) &= f(s) \quad \text{and} \quad F(s,1) = g(s), \\ F(0,t) &= a \qquad \text{and} \quad F(1,t) = b. \end{aligned}$$

In that case, one writes $f \simeq_p g$. The function F is called a **path homotopy** between f and g.

Example. Let f, g be paths in $\mathbb{R}^n$. The function $F : [0,1] \times [0,1] \to \mathbb{R}^n$ defined by

$$F(x, t) = (1-t)f(x) + tg(x)$$

is a path homotopy between f and g.

Proposition 7.2. *The relation $\simeq_p$ on paths in a topological space X with fixed origins and extremities is an equivalence relation.*

Proof. Given a path f, the function $F(x, t) = f(x)$ is the required path homotopy to get $f \simeq_p f$.

If $f \simeq_p g$ is established by a path homotopy $F(x, t)$, then $G(x, t) = F(x, 1-t)$ is a path homotopy between g and f.

Suppose that $f \simeq_p g$ by means of a path homotopy F, and $g \simeq_p h$ by means of a path homotopy G, then $f \simeq_p h$ by means of the path homotopy $H : [0,1] \times [0,1] \to X$ defined by the equation

$$H(x, t) = \begin{cases} F(x, 2t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ G(x, 2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

□

If f is a path, denote its path-homotopy equivalence class by $[f]$.

Definition 7.3. Let X be a topological space, f a path in X from a to b, and g a path in X from b to c. Define the product $f * g$ of f and g to be the path h in X given by the equation

$$h(s) = \begin{cases} f(2s) & \text{for } s \in \left[0, \frac{1}{2}\right], \\ g(2s-1) & \text{for } s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

The product operation of Definition 7.3 extends to an operation on path-homotopy classes defined by

$$[f]*[g] := [f*g].$$

Lemma 7.4. *Let X, Y be a topological space, $k : X \to Y$ a continuous function, and F is a path homotopy between two paths f, f' in X.*

- (i) *Then $k \circ F$ is a path homotopy in Y between $k \circ f$ and $k \circ f'$.*
- (ii) *Moreover, if g is a path in X with $f(1) = g(0)$, then $k \circ (f*g) = (k \circ f)*(k \circ g)$.*

Proof. (i) : The function $k \circ F : [0, 1] \times [0, 1] \to Y$ is continuous such that, if $s, t \in [0, 1]$,

$$k \circ F(s, 0) = k \circ f(s) \quad \text{and} \quad k \circ F(s, 1) = k \circ f'(s),$$
$$k \circ F(0, t) = k \circ f(0) = k \circ f'(0) \qquad \text{and} \qquad k \circ F(1, t) = k \circ f(1) = k \circ f'(1).$$

(ii) : We have

$$k \circ (f*g)(t) = k \circ \begin{cases} f(2t) & \text{for } t \in \left[0, \frac{1}{2}\right] \\ g(2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases} = \begin{cases} k \circ f(2t) & \text{for } t \in \left[0, \frac{1}{2}\right] \\ k \circ g(2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right] \end{cases} = (k \circ f)*(k \circ g)(t).$$

□

For $x \in X$, let e_x denote the constant path carrying all of $[0,1]$ to the point x. Given a path f in X from a to b, denote the reverse of f by $\bar{f}$. It is the path from b to a defined for $s \in [0, 1]$ by $\bar{f}(s) := f(1-s)$.

Proposition 7.5. *The operation $*$ on path-homotopy classes in a topological space X has the following properties:*

- (i) *If $[f]*\big([g]*[h]\big)$ is defined, so is $\big([f]*[g]\big)*[h]$, and they are equal.*
- (ii) *If f is a path in X from a to b, then*

$$[f]*[e_b] = [f] \quad \text{and} \quad [e_a]*[f] = [f].$$

- (iii) *If f is a path in X from a to b, then*

$$[f]*[\bar{f}] = [e_a] \quad \text{and} \quad [\bar{f}]*[f] = [e_b].$$

Proof. (ii) : If e_0 is the constant path at 0, and $i : [0, 1] \to [0, 1]$ the identity map, then $e_0 * i$ is a path from 0 to 1. Since i and $e_0 * i$ are paths in $\mathbb{R}$, there is a path homotopy F between them. Then $f \circ F$ is a path homotopy in X between the paths $f \circ i = f$ and $f \circ (e_0 * i) = (f \circ e_0)*(f \circ i) = e_a * f$. Similarly, using the fact that $i * e_1$ and i are path homotopic in $[0, 1]$, one shows that $[f]*[e_b] = [f]$.

(iii) : The path $i * \bar{i}$, that begins and ends at 0, is path homotopic to the constant path e_0 as paths in $\mathbb{R}$ once again. Denoting F a path homotopy between them, we get from Lemma 7.4 that $f \circ F$ is a path homotopy between $f \circ e_0 = e_a$ and $(f \circ i)*(f \circ \bar{i}) = f * \bar{f}$. With a similar argument, using the fact that $\bar{i} * i$ and e_1 are path homotopic in $[0, 1]$, one shows that $[\bar{f}]*[f] = [e_b]$.

(i) : We have

$$f*(g*h)(t) = \begin{cases} f(2t) & \text{for } t \in \left[0, \frac{1}{2}\right], \\ g*h(2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right], \end{cases} = \begin{cases} f(2t) & \text{for } t \in \left[0, \frac{1}{2}\right], \\ g\big(2(2t-1)\big) & \text{for } t \in \left[\frac{1}{2}, \frac{3}{4}\right], \\ h\big(2(2t-1)-1\big) & \text{for } t \in \left[\frac{3}{4}, 1\right], \end{cases}$$

$$\text{and} \quad (f*g)*h(t) = \begin{cases} f*g(2t) & \text{for } t \in \left[0, \frac{1}{2}\right], \\ h(2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right], \end{cases} = \begin{cases} f(4t) & \text{for } t \in \left[0, \frac{1}{4}\right], \\ g(4t-1) & \text{for } t \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ h(2t-1) & \text{for } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Then $\big(f*(g*h)\big) \circ \alpha = (f*g)*h$ with $\alpha : [0,\,1] \to [0,\,1]$ defined by $\alpha(s) = \begin{cases} 2s & \text{for } s \in \left[0, \frac{1}{4}\right] \\ s+\frac{1}{4} & \text{for } s \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ \frac{s}{2}+\frac{1}{2} & \text{for } s \in \left[\frac{1}{2}, 1\right] \end{cases}$.

As α and i are paths in $\mathbb{R}$, we get by Lemma 7.4 that $\big(f*(g*h)\big)\circ\alpha \simeq_p \big((f*g)*h\big)\circ i = (f*g)*h$. □

7.2 Fundamental Groups

Definition 7.6. Let X be a topological space, and $a \in X$. A path in X that starts and ends at a is called a **loop** at the **basepoint** a. The set of all homotopy classes $[f]$ of loops $f : [0,1] \to X$ at the basepoint a is denoted $\pi_1(X,\,a)$.

Proposition 7.7. *Let X be a topological space, and $a \in X$. The set $\pi_1(X,\,a)$ is a group with respect to the product $*$.*

Proof. By restricting to loops f, g with a fixed basepoint, we guarantee that the product $f*g$ or more exactly the product $[f]*[g] = [f*g]$ is defined. It remains to verify the three axioms for a group:

- From Proposition 7.5 (i), for all $[f],[g],[h] \in \pi_1(X,\,a)$, $[f]*\big([g]*[h]\big) = \big([f]*[g]\big)*[h]$.
- From Proposition 7.5 (ii), for every $[f] \in \pi_1(X,\,a)$, $[f]*[e_a] = [f]$ and $[e_a]*[f] = [f]$.
- From Proposition 7.5 (iii), for every $[f] \in \pi_1(X,\,a)$, $[f]*[\bar{f}] = [e_a]$ and $[\bar{f}]*[f] = [e_a]$.

□

Definition 7.8. Let X be a topological space, and $a \in X$. The group $\pi_1(X,\,a)$ is called the **fundamental group** of X at the basepoint a.

Example. For a convex set X in $\mathbb{R}^n$ with basepoint $a \in X$, $\pi_1(X,\,a)$ is the trivial one-element group. Indeed the function $F : [0,1]\times[0,1] \to \mathbb{R}^n$ defined by

$$F(x,t) = (1-t)f(x) + tg(x)$$

is a path homotopy between any loops f, g based at a.

Definition 7.9. A topological space X is said to be **simply connected** if it is a path connected space and if $\pi_1(X,\,a)$ is the trivial one-element group for every $a \in X$.

Proposition 7.10. *Let X be a simply connected topological space. Then, any paths in X having the same origin and extremity are path homotopic.*

Proof. Let f, g be paths in X from a to b. Then $f*\bar{g}$ is defined and is a loop on X based at a. Since X is simply connected, $f*g$ is path homotopic to e_a. Using Proposition 7.5, we get

$$[f] = [f]*[e_b] = [f]*[\bar{g}*g] = [f*\bar{g}]*[g] = [e_a]*[g] = [g].$$

□

Proposition 7.11. *Let X be a topological space, $a, b \in X$, and f a path from a to b. Define the map $\hat{f} : \pi_1(X, a) \to \pi_1(X, b)$ by*

$$\hat{f}\big([h]\big) := [\bar{f}] * [h] * [f].$$

Then the map $\hat{f}$ is a group isomorphism.

Proof. Let $[g], [h] \in \pi_1(X, a)$. We have

$$\begin{aligned}\hat{f}\big([g]\big) * \hat{f}\big([h]\big) &= \big([\bar{f}] * [g] * [f]\big) * \big([\bar{f}] * [h] * [f]\big)\\ &= [\bar{f}] * [g] * [h] * [f]\\ &= \hat{f}\big([g] * [h]\big).\end{aligned}$$

Then, $\hat{f}$ is a homomorphism. To prove that $\hat{f}$ is an isomorphism, we show that $\widehat{\bar{f}} : \pi_1(X, b) \to \pi_1(X, a)$ defined for every $[h] \in \pi_1(X, b)$ by

$$\widehat{\bar{f}}\big([h]\big) := [f] * [h] * [\bar{f}]$$

is an inverse for $\hat{f}$. We have $\widehat{\bar{f}} \circ \hat{f}\big([h]\big) = [f] * \big([\bar{f}] * [h] * [f]\big) * [\bar{f}] = [h]$. A similar computation shows that $\hat{f} \circ \widehat{\bar{f}}\big([h]\big) = [h]$. □

Suppose that $h : X \to Y$ is a continuous function that carries the point a of X to the point b of Y. One denotes this fact by writing $h : (X, a) \to (Y, b)$.

Definition 7.12. Let X, Y be topological spaces, and $h : (X, a) \to (Y, b)$ a continuous function. Define $h_* : \pi_1(X, a) \to \pi_1(Y, b)$ by

$$h_*\big([f]\big) := [h \circ f].$$

The map h_* is called the **homomorphism induced** by h relative to the basepoint a.

Proposition 7.13. *Let X, Y, Z be topological spaces.*

(i) *If $h : (X, a) \to (Y, b)$ and $k : (Y, b) \to (Z, c)$ are continuous maps, then $(k \circ h)_* = k_* \circ h_*$.*

(ii) *If $i : (X, a) \to (X, a)$ is the identity map, then i_* is the identity homomorphism.*

Proof. (i) : We have both equalities

$$\begin{aligned}&(k \circ h)_*\big([f]\big) = \big[(k \circ h) \circ f\big],\\ &(k_* \circ h_*)\big([f]\big) = k_*\Big(h_*\big([f]\big)\Big) = k_*\big([h \circ f]\big) = \big[k \circ (h \circ f)\big].\end{aligned}$$

(ii) : We have $i_*\big([f]\big) = [i \circ f] = [f]$. □

Corollary 7.14. *Let X, Y be topological spaces. If $h : (X, a) \to (Y, b)$ is a homeomorphism from X to Y, then h_* is an isomorphism from $\pi_1(X, a)$ to $\pi_1(Y, b)$.*

Proof. Let $k : (Y, b) \to (X, a)$ be the inverse of h. Then $k_* \circ h_* = (k \circ h)_* = i_*$, where i is the identity map of (X, a). Besides, $h_* \circ k_* = (h \circ k)_* = j_*$, where j is the identity map of (Y, b). As i_* and j_* are the identity homomorphisms of $\pi_1(X, a)$ and $\pi_1(Y, b)$ respectively, k_* is then the inverse of h_*. □

Proposition 7.15. *Let X, Y be topological spaces, and $(a, b) \in X \times Y$. Then $\pi_1\big(X \times Y, (a, b)\big)$ is isomorphic to $\pi_1(X, a) \times \pi_1(Y, b)$.*

Proof. We know from Proposition 3.14 that the existence of a loop $f:[0,\,1]\to X\times Y$ at the basepoint $(a,\,b)$ is equivalent to the existence of a loop $g:[0,\,1]\to X$ at the basepoint a, and a loop $h:[0,\,1]\to Y$ at the basepoint b such that $f=(g,\,h)$. We also know from Proposition 3.14 that the existence of a path homotopy $F:[0,\,1]\times[0,\,1]\to X\times Y$ between two loops f_1,f_2 at the basepoint $(a,\,b)$ is equivalent to the existence of a path homotopy $G:[0,\,1]\times[0,\,1]\to X$ between two loops g_1,g_2 at the basepoint a, and a path homotopy $H:[0,\,1]\times[0,\,1]\to Y$ between two loops h_1,h_2 at the basepoint b such that $f_1=(g_1,\,h_1)$, $f_2=(g_2,\,h_2)$, and $F=(G,\,H)$. Thus, the function $\alpha:\pi_1\big(X\times Y,\,(a,\,b)\big)\to\pi_1(X,\,a)\times\pi_1(Y,\,b)$ defined, for a loop $f=(g,\,h)$ at the basepoint $(a,\,b)$, by $\alpha\big([f]\big)=\big([g],\,[h]\big)$ is bijective. It can also be extended to a group homomorphism since, for two loops $f_1=(g_1,\,h_1)$, $f_2=(g_2,\,h_2)$ at the basepoint $(a,\,b)$, we have

$$\alpha\big([f_1]*[f_2]\big)=\alpha\big([f_1*f_2]\big)=\big([g_1*g_2],\,[h_1*h_2]\big)=\big([g_1]*[g_2],\,[h_1]*[h_2]\big)=\alpha\big([f_1]\big)*\alpha\big([f_2]\big).$$

Hence, α is an isomorphism. □

7.3 The Fundamental Group of $\mathbb{S}^n$

Lemma 7.16. *For $p_1,p_2,p_3\in\mathbb{R}^n$, the triangle of vertices p_1,p_2,p_3 is*

$$T=\{t_1p_1+t_2p_2+t_3p_3\mid t_1,t_2,t_3\in\mathbb{R}_+,\,t_1+t_2+t_3=1\}.$$

Consider a topological space X, and a continuous function $f:T\to X$. For $i,j\in\{1,2,3\}$ with $i<j$, the standard parametrisation of f restricted to the edge from p_i to p_j is the path

$$f_{ij}:[0,\,1]\to X,\quad t\mapsto f\big((1-t)p_i+tp_j\big)$$

from $f(p_i)$ to $f(p_j)$. We have, $f_{13}\simeq_p f_{12}*f_{23}$.

Proof. Consider the function

$$q:[0,\,1]\times[0,\,1]\to T,\quad (t,\,s)\mapsto\begin{cases}(1-t-ts)p_1+2tsp_2+(t-ts)p_3 & \text{for } t\le\frac{1}{2},\\ (1-t-s-ts)p_1+2(1-t)sp_2+(t-s+ts)p_3 & \text{for } t\ge\frac{1}{2}.\end{cases}$$

We have

$$f\big(q(t,\,0)\big)=\begin{cases}f\big((1-t)p_1+tp_3\big)=f_{13}(t) & \text{for } t\le\frac{1}{2}\\ f\big((1-t)p_1+tp_3\big)=f_{13}(t) & \text{for } t\ge\frac{1}{2}\end{cases}=f_{13}(t),$$

$$f\big(q(t,\,1)\big)=\begin{cases}f\big((1-2t)p_1+2tp_2\big)=f_{12}(2t) & \text{for } t\le\frac{1}{2}\\ f\Big(\big(1-(2t-1)\big)p_2+(2t-1)p_3\Big)=f_{23}(2t-1) & \text{for } t\ge\frac{1}{2}\end{cases}=f_{12}*f_{23}(t),$$

$$f\big(q(0,\,s)\big)=f(p_1)\quad\text{and}\quad f\big(q(1,\,s)\big)=f(p_3).$$

Hence, the function

$$F:[0,\,1]\times[0,\,1]\to X,\quad (t,\,s)\mapsto f\big(q(t,\,s)\big)$$

is a path homotopy from f_{13} to $f_{12}*f_{23}$. □

Lemma 7.17. *Let X be a topological space, $f : [0, 1] \to X$ a path in X, and $a_0, \dots, a_n \in \mathbb{R}$ such that $0 = a_0 < a_1 < \cdots < a_n = 1$. For $i \in \{1, \dots, n\}$, let $l_i : [0, 1] \to [a_{i-i}, a_i]$ be the affine function such that $l_i(0) = a_{i-1}$ and $l_i(1) = a_i$, and*

$$f_i : [0, 1] \to X, \quad t \mapsto f \circ l_i(t)$$

the standard parametrisation of f restricted to $[a_{i-i}, a_i]$. Then, $[f] = [f_1] * \cdots * [f_n]$.

Proof. Using Lemma 7.16 with f equal to the identity map $i_{[a_0, a_2]}$ on $[a_0, a_2]$, we prove that $l_1 * l_2 \simeq_p l_{12}$ which is the affine function such that $l_{12}(0) = a_0$ and $l_{12}(1) = a_2$. More generally, for $k \in \{3, \dots, n\}$, we can use Lemma 7.16 with f equal to the identity map $i_{[a_0, a_k]}$ on $[a_0, a_k]$ to prove that $l_{1\,k-1} * l_k \simeq_p l_{1k}$, where $l_{1\,k-1}$ and l_{1k} are the affine functions such that $l_{1\,k-1}(0) = l_{1k} = a_0$, $l_{1\,k-1}(1) = a_{k-1}$, and $l_{1k} = a_k$. Hence, we successively obtain

$$\begin{aligned} l_1 * l_2 * l_3 * \cdots * l_n &= l_{12} * l_3 * \cdots * l_n \\ &= l_{13} * \cdots * l_n \\ &= l_{1n} \end{aligned}$$

which is the identity map on $[0, 1]$. We deduce from Lemma 7.4 that

$$\begin{aligned} &(f \circ l_1) * (f \circ l_2) * (f \circ l_3) * \cdots * (f \circ l_n) = f \circ l_{1n} = f \\ &f_1 * f_2 * f_3 * \cdots * f_n = f \\ &[f_1] * [f_2] * [f_3] * \cdots * [f_n] = [f]. \end{aligned}$$

□

Proposition 7.18. *Let X be topological space, and A, B two open subsets of X such that $X = A \cup B$ and $A \cap B \neq \varnothing$. Suppose that A, B are path connected, and take $x \in A \cap B$. Consider the inclusion maps $i : A \hookrightarrow X$ and $j : B \hookrightarrow X$. Then, $\pi_1(X, x)$ is generated by the images of the induced homomorphisms*

$$i_* : \pi_1(A, x) \to \pi_1(X, x) \quad \text{and} \quad j_* : \pi_1(B, x) \to \pi_1(X, x).$$

Proof. Let $f : [0, 1] \to X$ be a loop based at x. We know from Theorem 8.10 that there exists a positive integer n such that, for every $i \in \{1, \dots, n\}$, the restriction of f to the interval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ is contained in A or in B. Let f_i be the standard parametrisation of f restricted to $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, that is

$$f_i : [0, 1] \to A \text{ (or } B), \quad t \mapsto f\Big(\frac{i-1+t}{n}\Big).$$

Since A, B are path connected, we can find a path h_i from $f\Big(\frac{i}{n}\Big)$ to x so that

- if $f\Big(\frac{i}{n}\Big) \in A$, then $h_i : [0, 1] \to A$ is a path in A,
- if $f\Big(\frac{i}{n}\Big) \in B$, then $h_i : [0, 1] \to A$ is a path in B.

Using Lemma 7.17, we may write

$$\begin{aligned} f &= f_1 * f_2 * \cdots * f_i * \cdots * f_{n-1} * f_n \\ &= f_1 * h_1 * \bar{h}_1 * f_2 * h_2 * \cdots * \bar{h}_{i-1} * f_i * h_i * \cdots * \bar{h}_{n-2} * f_{n-1} * h_{n-1} * \bar{h}_{n-1} * f_n \\ &= k_1 * k_2 * \cdots * k_{n-1} * k_n, \end{aligned}$$

where

$$k_1 = f_1 * h_1,\ k_2 = \bar{h}_1 * f_2 * h_2,\ \ldots,\ k_i = \bar{h}_{i-1} * f_i * h_i,\ \ldots,\ k_{n-1} = \bar{h}_{n-2} * f_{n-1} * h_{n-1},\ k_n = \bar{h}_{n-1} * f_n.$$

To finish, for every $i \in \{1, \ldots, n\}$, k_i is a loop based at x in A or in B. □

Corollary 7.19. *Let X be a topological space, and A, B open sets of X such that $X = A \cup B$ and $A \cap B \neq \varnothing$. If A and B are simply connected, then X is simply connected.*

Proof. As A and B are path connected, we deduce from Proposition 5.18 that X is path connected. Choose a base point $x \in A \cap B$. Since $\pi_1(A, x)$ and $\pi_1(B, x)$ are the trivial one-element group, $\pi_1(X, x)$ is then generated by the neutral element by Proposition 7.18, so it is trivial. □

Corollary 7.20. *If n is a positive integer such that $n \geq 2$, then $\mathbb{S}^n$ is simply connected.*

Proof. Write $\mathbb{S}^n = A \cup B$, where $A = \mathbb{S}^n \setminus \big\{(0, \ldots, 0, 1)\big\}$ and $B = \mathbb{S}^n \setminus \big\{(0, \ldots, 0, -1)\big\}$. We know from the stereographic projection of A onto $\mathbb{R}^n$ that A is homeomorphic to $\mathbb{R}^n$. Moreover, the function $f : A \to B,\ a \mapsto -a$ is a homeomorphism between A and B. Hence, A and B are simply connected, and also $\mathbb{S}^n$ by Corollary 7.19. □

Chapter 8

Covering Spaces

8.1 Covering Maps

Definition 8.1. Let X,Y be topological spaces, and $p : X \to Y$ a continuous surjective function. An open set A of Y is said to be **evenly covered** by p if the inverse image $p^{-1}(A)$ is equal to $\bigsqcup_{i\in I} A_i$ such that A_i is an open subset of X, and the restriction of p to A_i is a homeomorphism of A_i to A. The family $\{A_i\}_{i\in I}$ is called a partition of $p^{-1}(A)$ into **slices**.

Definition 8.2. Let X,Y be open topological spaces, and $p : X \to Y$ a continuous surjective function. If every point a of Y has an open neighborhood A that is evenly covered by p, then p is called a **covering map**, and X is said to be a **covering space** of Y.

Example. Consider $\mathbb{R}$ with the usual topology, and $\mathbb{S}^1 = \big\{(\cos t, \sin t) \bigm| t \in [0, 2\pi]\big\}$ equipped with the topology induced by the usual topology of $\mathbb{R}^2$. For any point $a = (\cos u, \sin u) \in \mathbb{S}^1$, the set $U_a = \big\{(\cos t, \sin t) \bigm| t \in (u-1, u+1)\big\}$ is then an open neighborhood of a. The function $p : \mathbb{R} \to \mathbb{S}^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ is continuous and surjective. Moreover,

- we have $p^{-1}(U_a) = \bigsqcup_{k\in\mathbb{Z}} \Big(\dfrac{u-1}{2\pi} + k, \dfrac{u+1}{2\pi} + k\Big)$, where $\Big(\dfrac{u-1}{2\pi} + k, \dfrac{u+1}{2\pi} + k\Big)$ is open in $\mathbb{R}$,
- the restriction p_k of p to $\Big(\dfrac{u-1}{2\pi} + k, \dfrac{u+1}{2\pi} + k\Big)$ is clearly a homeomorphism onto U_a.

Then, p is a covering map.

Definition 8.3. Let X,Y be topological spaces, and $f : X \to Y$ a function. A function $s : Y \to X$ is called a **section** of f is $p\big(s(y)\big) = y$ for every $y \in Y$.

Proposition 8.4. *Let X,Y be topological spaces, and $p : X \to Y$ a covering map. For every evenly covered set $V \subseteq Y$, and every point $x \in p^{-1}(V)$, there exists a continuous section $s : V \to p^{-1}(V)$ of the restriction $p : p^{-1}(V) \to V$ such that $s\big(p(x)\big) = x$. If V is connected, then s is unique.*

Proof. We can write $p^{-1}(V) = U \sqcup W$ such that U and W are open, $x \in U$, and the restriction $p_{|U} : U \to V$ is a homeomorphism. The inverse $s = p_{|U}^{-1}$ is clearly a continuous section of $p_{|U}$, and consequently of p by extending its codomain to $p^{-1}(V)$.
If V is connected, then U is connected and is a connected component of $p^{-1}(V)$. Suppose $r : V \to X$ is another continuous section of p such that $r\big(p(x)\big) = x$. Since $r(V) \subseteq p^{-1}(V)$ and V is connected, then

$r(V)$ is contained in the connected component of $p^{-1}(V)$ that contains x which is U. As $p\big(r(y)\big) = y$ for every $y \in V$, $r : V \to U$ is then the inverse of $p_{|U} : U \to V$. □

Proposition 8.5. *Let X, Y be topological spaces, and $p : X \to Y$ a covering map. If Y_0 is a subspace of Y, and if $X_0 = p^{-1}(Y_0)$, then the map $p_0 : X_0 \to Y_0$ obtained by restricting p is a covering map.*

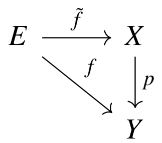
Proof. Given $y \in Y_0$, let V be an open set in Y containing y that is evenly covered by p. If $\{U_i\}_{i \in I}$ is a partition of $p^{-1}(V)$ into slices, then $V \cap Y_0$ is a neighborhood of y in Y_0, and $\{U_i \cap X_0\}_{i \in I}$ is formed by disjoint open sets in X_0 whose union is $p^{-1}(V \cap Y_0)$. Moreover, the restriction of p to $U_i \cap X_0$ is a homeomorphism onto $V \cap Y_0$. □

Proposition 8.6. *Let X, X', Y, Y' be topological spaces, and $p : X \to Y$, $p' : X' \to Y'$ covering maps. Then $p \times p' : X \times X' \to Y \times Y'$ is a covering map.*

Proof. Let $(y, y') \in Y \times Y'$, and V, V' neighborhoods of y, y' respectively, that are evenly covered by p, p' respectively. Let $\{U_i\}_{i \in I}, \{U'_j\}_{j \in J}$ be partitions into slices of $p^{-1}(V), p'^{-1}(V')$ respectively. Then $(p \times p')^{-1}(V \times V') = \bigsqcup_{\substack{i \in I \\ j \in J}} U_i \times U'_j$. Moreover, the restriction of $p \times p'$ to $U_i \times U'_j$ is a homeomorphism onto $V \times V'$. □

8.2 Function Liftings

Definition 8.7. Let E, X, Y be topological spaces, $p : X \to Y$ a covering map, and $f : E \to Y$ a continuous function. A **lifting** of f is a function $\tilde{f} : E \to X$ such that $p \circ \tilde{f} = f$.



Example. Consider the covering map $p : \mathbb{R} \to \mathbb{S}^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. The path $f : [0, 1] \to \mathbb{S}^1$ from $(1, 0)$ to $(-1, 0)$ given by $f(t) = (\cos \pi t, \sin \pi t)$ lifts to the path $\tilde{f} : [0, 1] \to \mathbb{R}$ from 0 to $\frac{1}{2}$ given by $\tilde{f}(t) = \frac{t}{2}$. The path $g : [0, 1] \to \mathbb{S}^1$ given by $g(t) = (\cos \pi t, -\sin \pi t)$ from $(1, 0)$ to $(-1, 0)$ lifts to the path $\tilde{g} : [0, 1] \to \mathbb{R}$ from 0 to $-\frac{1}{2}$ given by $\tilde{g}(t) = -\frac{t}{2}$.

Lemma 8.8. *Let X, Y be topological spaces, and $p : X \to Y$ a covering map. Consider the subspace*

$$X \times_p X = \big\{(a, b) \in X \times X \bigm| p(a) = p(b)\big\}$$

of the product space $X \times X$. Then, $\Delta = \big\{(a, a) \bigm| a \in X\big\}$ is an open and a closed subset of $X \times_p X$.

Proof. Take $(x, x) \in \Delta$ and choose an open set $U \subseteq X$ such that $x \in U$ and the restriction $p : U \to Y$ is injective. Then, $(U \times U) \cap (X \times_p X) = U \times_p U$ is an open neighborhood of (x, x) in $X \times_p X$. As $U \times_p U = \big\{(a, b) \in U \times U \bigm| p(a) = p(b)\big\} = \big\{(a, a) \bigm| a \in U\big\} \subseteq \Delta$, then Δ is a neighborhood of points, so is open in $X \times_p p$ by Proposition 1.9.

Take $(x_1, x_2) \in X \times_p X \setminus \Delta$, and choose an evenly covered open set $V \subseteq Y$ containing $p(x_1) = p(x_2)$. Since $x_1 \neq x_2$, they cannot be in the same slice, so there exist disjoint open sets $U_1, U_2 \in p^{-1}(V)$ such that $x_1 \in U_1$ and $x_2 \in U_2$. Therefore, the set $(U_1 \times U_2) \cap (X \times_p X)$ contains (x_1, x_2), is open in $X \times_p X$, and is included in $X \times_p X \setminus \Delta$. We deduce from Proposition 1.9 that $X \times_p X \setminus \Delta$ is open, so Δ is closed in $X \times_p X$. □

Lemma 8.9. *Let X,Y be topological spaces, $p: X \to Y$ a covering map, E a connected space, and $f: E \to Y$ a continuous function. If $g: E \to X$ and $h: E \to X$ are two liftings of f, we have either $g = h$ or $g(e) \neq h(e)$ for every $e \in E$.*

Proof. Recall that $X \times_Y X = \big\{(a,b) \in X \times X \;\big|\; p(a) = p(b)\big\}$ and $\Delta = \big\{(a,a) \;\big|\; a \in X\big\}$. Consider the continuous function $\Phi : E \to X \times_Y X$ defined by $\Phi(e) = \big(g(e), h(e)\big)$. Let $A = \big\{e \in E \;\big|\; g(e) = h(e)\big\} = \Phi^{-1}(\Delta)$. We know from Lemma 8.8 that Δ is open and closed in $X \times_p X$. Then, A is open and closed in E. Since E is connected, either $A = E$ or $A = \varnothing$. □

Theorem 8.10 (Lebesgue number)**.** *Let X be a compact metric space with metric d, Y a topological space, $\mathscr{O}$ a family of open sets covering Y, and $f: X \to Y$ a continuous function. There exists $\rho \in \mathbb{R}_+^*$ such that, for any $x \in X$, $f\big(B(x,\rho)\big)$ is contained in an open set of $\mathscr{O}$.*

Proof. For any $n \in \mathbb{N}$, let X_n be the set of points $x \in X$ having the property that there exists $U \in \mathscr{O}$ such that $B(x, 2^{-n}) \subseteq f^{-1}(U)$. For any $x \in X$, there exists $U \in \mathscr{O}$ such that $x \in f^{-1}(U)$. As $f^{-1}(U)$ is open, there exists $n \in \mathbb{N}$ such that $B(x, 2^{-n}) \subseteq f^{-1}(U)$, then $\bigcup_{n \in \mathbb{N}} X_n = X$.

It is clear that $X_n \subseteq X_{n+1}$. Moreover, $X_n \subseteq X_{n+1}^{\circ}$. Indeed, let $x \in X_n$ and $U \in \mathscr{O}$ such that $B(x, 2^{-n}) \subseteq f^{-1}(U)$. For every $z \in X$ such that $d(x,z) < 2^{-n-1}$, we have $B(z, 2^{-n-1}) \subseteq B(x, 2^{-n}) \subseteq f^{-1}(U)$, then $z \in X_{n+1}$. Hence $B(x, 2^{-n-1}) \subseteq X_{n+1}$, meaning that X_{n+1} is a neighborhood of x.
The fact $X_n \subseteq X_{n+1}^{\circ}$ implies $\bigcup_{n \in \mathbb{N}} X_n \subseteq \bigcup_{n \in \mathbb{N}} X_n^{\circ}$, and then $\bigcup_{n \in \mathbb{N}} X_n^{\circ} = X$. As X is compact, $X = X_n^{\circ}$ for some $n \in \mathbb{N}$, and consequently $X = X_n$. □

Theorem 8.11. *Let X,Y be topological spaces, $p: X \to Y$ a covering map, and $(a,b) \in X \times Y$ such that $p(a) = b$. Any path $f: [0,1] \to Y$ beginning at b has a unique lifting to a path $\tilde{f} : [0,1] \to X$ beginning at a.*

Proof. We know from Lemma 8.9 there exists at most one lifting $\tilde{f} : [0,1] \to X$ such that $\tilde{f}(0) = a$. Then, the existence remains. Let $\mathscr{O}$ be a family of evenly covered open sets covering Y. We know from Theorem 8.10 that there exist $n \in \mathbb{N}$ and $V_1, \dots, V_n \in \mathscr{O}$ such that $f\Big(\Big[\frac{i-1}{n}, \frac{i}{n}\Big]\Big) \subseteq V_i$ for every $i \in \{1, \dots, n\}$. We recursively define n continuous functions $g_i : \Big[\frac{i-1}{n}, \frac{i}{n}\Big] \to X$ for every $i \in \{1, \dots, n\}$ such that

- $\forall t \in \Big[\frac{i-1}{n}, \frac{i}{n}\Big],\ p\big(g_i(t)\big) = f(t)$,
- $g_1(0) = a$, and $g_i\Big(\frac{i}{n}\Big) = g_{i+1}\Big(\frac{i}{n}\Big)$.

Using Proposition 8.4, we deduce the existence of a section $s_1 : V_1 \to p^{-1}(V_1)$ of the restriction $p : p^{-1}(V_1) \to V_1$ such that $s_1\big(p(a)\big) = a$. Then, we may define $g_1 : \Big[0, \frac{1}{n}\Big] \to X$ by $g_1(t) = s_1\big(f(t)\big)$. Suppose that g_i has already been defined, and consider a section $s_{i+1} : V_{i+1} \to p^{-1}(V_{i+1})$ of the restriction $p : p^{-1}(V_{i+1}) \to V_{i+1}$ such that $s_{i+1}\bigg(f\Big(\frac{i}{n}\Big)\bigg) = s_{i+1}\bigg(p\bigg(g_i\Big(\frac{i}{n}\Big)\bigg)\bigg) = g_i\Big(\frac{i}{n}\Big)$. We may define $g_{i+1} : \Big[\frac{i}{n}, \frac{i+1}{n}\Big] \to X$ by $g_{i+1}(t) = s_{i+1}\big(f(t)\big)$. Hence $g_1 * g_2 * \cdots * g_n$ is the required lifting $\tilde{f}$. □

Proposition 8.12. *Let X, Y be topological spaces, $p : X \to Y$ a covering map, and $(a, b) \in X \times Y$ such that $p(a) = b$. Consider a continuous function $F : [0, 1] \times [0, 1] \to Y$ such that $F(0, 0) = b$. There exists a unique lifting of F to a continuous function*

$$\tilde{F} : [0, 1] \times [0, 1] \to X \quad \textit{such that} \quad \tilde{F}(0, 0) = a.$$

Proof. We know from Lemma 8.9 there exists at most one lifting $\tilde{F} : [0, 1] \times [0, 1] \to X$ such that $\tilde{F}(0, 0) = a$. Then, the existence remains.

Let $\mathscr{O}$ be a family of evenly covered open sets covering Y. We know from Theorem 8.10 that there exist $m, n \in \mathbb{N}$ and $V_{11}, \dots, V_{mn} \in \mathscr{O}$ such that $F\Big(\Big[\frac{i-1}{m}, \frac{i}{m}\Big] \times \Big[\frac{j-1}{n}, \frac{j}{n}\Big]\Big) \subseteq V_{ij}$ for every $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$. We recursively define on each row and from the bottom to the top mn continuous functions $\tilde{F}_{ij} : \Big[\frac{i-1}{m}, \frac{i}{m}\Big] \times \Big[\frac{j-1}{n}, \frac{j}{n}\Big] \to X$ for every $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$ such that

- $\forall (s, t) \in \Big[\frac{i-1}{n}, \frac{i}{n}\Big] \times \Big[\frac{j-1}{n}, \frac{j}{n}\Big]$, $p\big(\tilde{F}_{ij}(s, t)\big) = F(s, t)$,
- $\tilde{F}_{11}(0, 0) = a$ and $\tilde{F}_{i1}\Big(\frac{i}{m}, 0\Big) = \tilde{F}_{i+1\,1}\Big(\frac{i}{m}, 0\Big)$,
- $\tilde{F}_{1\,j+1}\Big(0, \frac{j}{n}\Big) = \tilde{F}_{1\,j}\Big(0, \frac{j}{n}\Big)$ and $\tilde{F}_{i\,1+1}\Big(\frac{i}{m}, \frac{j}{n}\Big) = \tilde{F}_{i+1\,j+1}\Big(\frac{i}{m}, \frac{j}{n}\Big)$.

Using Proposition 8.4, we deduce the existence of a section $s_{11} : V_{11} \to p^{-1}(V_{11})$ of the restriction $p : p^{-1}(V_{11}) \to V_{11}$ such that $s_{11}\big(p(a)\big) = a$. Then, we may define $\tilde{F}_{11} : \Big[0, \frac{1}{m}\Big] \times \Big[0, \frac{1}{n}\Big] \to X$ by $\tilde{F}_{11}(s, t) = s_{11}\big(F(s, t)\big)$. Suppose that $\tilde{F}_{11}, \dots, \tilde{F}_{ij}$ have already been defined, and consider a section $s_{i+1,j} : V_{i+1,j} \to p^{-1}(V_{i+1,j})$ of the restriction $p : p^{-1}(V_{i+1,j}) \to V_{i+1,j}$ such that

$$s_{i+1,j}\left(F\Big(\frac{i}{m}, \frac{j}{n}\Big)\right) = s_{i+1,j}\left(p\left(\tilde{F}_{ij}\Big(\frac{i}{m}, \frac{j}{n}\Big)\right)\right) = \tilde{F}_{ij}\Big(\frac{i}{m}, \frac{j}{n}\Big).$$

We may define $\tilde{F}_{i+1,j} : \Big[\frac{i}{m}, \frac{i+1}{m}\Big] \times \Big[\frac{j}{n}, \frac{j+1}{n}\Big] \to X$ by $\tilde{F}_{i+1,j} = s_{i+1}\big(F(s, t)\big)$.

Remark that, due to the uniqueness of the lifting of the path $F\Big(\frac{i}{m}, \frac{j-1+t}{n}\Big)$ with variable t beginning at $\tilde{F}_{ij}\Big(\frac{i}{m}, \frac{j-1}{n}\Big) = \tilde{F}_{i+1\,j}\Big(\frac{i}{m}, \frac{j-1}{n}\Big)$, we have

$$\forall (s, t) \in \Big\{\frac{i}{m}\Big\} \times \Big[\frac{j-1}{n}, \frac{j}{n}\Big],\ \tilde{F}_{ij}(s, t) = \tilde{F}_{i+1\,j}(s, t).$$

Using the same argument with the lifting beginning at $\tilde{F}_{ij}\Big(\frac{i}{m}, \frac{j}{n}\Big) = \tilde{F}_{i\,j+1}\Big(\frac{i}{m}, \frac{j}{n}\Big)$, we get

$$\forall (s, t) \in \Big[\frac{i-1}{m}, \frac{i}{m}\Big] \times \Big\{\frac{j}{n}\Big\},\ \tilde{F}_{ij}(s, t) = \tilde{F}_{i\,j+1}(s, t).$$

Hence, $\tilde{F} = \tilde{F}_{ij}$ on $\Big[\frac{i-1}{m}, \frac{i}{m}\Big] \times \Big[\frac{j-1}{n}, \frac{j}{n}\Big] \to X$, for every $(i, j) \in \{1, \dots, m\} \times \{1, \dots, n\}$, is the required lifting of F. □

Corollary 8.13. *Let X,Y be topological spaces, $p : X \to Y$ a covering map, and $(a, b) \in X \times Y$ such that $p(a) = b$. Consider two paths $f : [0, 1] \to Y$ and $g : [0, 1] \to Y$ beginning at b and ending c, and their respective liftings $\tilde{f}$ and $\tilde{g}$ beginning at a. The following conditions are equivalent:*

- (i) *f and g are path homotopic,*
- (ii) *$\tilde{f}(1) = \tilde{g}(1)$ and $\tilde{f}, \tilde{g}$ are path homotopic.*

Proof. $(i) \Rightarrow (ii)$: Consider a path homotopy $F : [0, 1] \times [0, 1] \to Y$ such that $F(0, t) = f(t)$, $F(1, t) = g(t)$, $F(s, 0) = b$, and $F(s, 1) = c$. Let $\tilde{F} : [0, 1] \times [0, 1] \to X$ the lifting of F such that $\tilde{F}(0, 0) = a$ described in Proposition 8.12. Path lifting uniqueness implies $\tilde{F}(0, t) = \tilde{f}(t)$ and $\tilde{F}(1, t) = \tilde{g}(t)$. Moreover, $\tilde{F}(s, 0)$ and $\tilde{F}(s, 1)$ are the liftings of e_b and e_c respectively, so must be constant. Consequently, $\tilde{f}(1) = \tilde{g}(1)$ and $\tilde{F}$ is a path homotopy between $\tilde{f}$ and $\tilde{g}$.
$(ii) \Rightarrow (i)$: If $\tilde{f}$ and $\tilde{g}$ are path homotopic with path homotopy $\tilde{F}$, then $p \circ \tilde{f} = f$ and $p \circ \tilde{g} = g$ are path homotopic with path homotopy $p \circ \tilde{F}$. □

Definition 8.14. Let X,Y be topological spaces, and $p : X \to Y$ a covering map. Let $b \in Y$ and choose $a \in X$ so that $p(a) = b$. Given an element $[f]$ of $\pi_1(Y,b)$, let $\tilde{f} : [0,1] \to X$ be the lifting of f to a path in X that begins at a. Define the function

$$\phi : \pi_1(Y,b) \to p^{-1}(b), \quad [f] \mapsto \tilde{f}(1).$$

One calls ϕ the **lifting correspondence** derived from the covering map p and the origin a.

Proposition 8.15. *Let X,Y be topological spaces, and $p : X \to Y$ a covering map. Let $b \in Y$ and choose $a \in X$ so that $p(a) = b$. If X is path connected, then the lifting correspondence*

$$\phi : \pi_1(Y,b) \to p^{-1}(b), \quad [f] \mapsto \tilde{f}(1)$$

is surjective. If X is simply connected, then ϕ is bijective.

Proof. Let $a' \in p^{-1}(b)$, and $\tilde{f} : [0, 1] \to X$ a path from a to a'. The path $\tilde{f}$ is the lifting of $f = p \circ \tilde{f}$ which is a loop in Y at b, then $\phi\big([f]\big) = a'$, and ϕ is consequently surjective.
Suppose that X is simply connected. Take $[f], [g] \in \pi_1(Y,b)$ such that $\phi\big([f]\big) = \phi\big([g]\big)$. Let $\tilde{f}$ and $\tilde{g}$ be the liftings of f and g respectively that begin at a. Then $\tilde{f}(1) = \tilde{g}(1)$. The fact X is simply connected implies the existence of a path homotopy $\tilde{F}$ between $\tilde{f}$ and $\tilde{g}$. Then $p \circ \tilde{F}$ is path homotopy between f and g, that is $[f] = [g]$. □

Theorem 8.16. *The group $\pi_1\big(\mathbb{S}^1, (1, 0)\big)$ is isomorphic to the additive group $(\mathbb{Z}, +)$.*

Proof. Consider the covering map $p : \mathbb{R} \to \mathbb{S}^1$ given by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. We have $p^{-1}\big((1, 0)\big) = \mathbb{Z}$. Since $\mathbb{R}$ is simply connected, we deduce from Proposition 8.15 that the lifting correspondence

$$\phi : \pi_1\big(\mathbb{S}^1, (1, 0)\big) \to \mathbb{Z}, \quad [f] \mapsto \tilde{f}(1)$$

is bijective. It remains to show that ϕ is a homeomorphism.
Given $[f], [g] \in \pi_1\big(\mathbb{S}^1, (1, 0)\big)$, let $\tilde{f}, \tilde{g}$ be their respective liftings to paths in $\mathbb{R}$ beginning at 0. Denote $n = \tilde{f}(1)$ and $m = \tilde{g}(1)$. Define the path

$$\tilde{\tilde{g}} : [0, 1] \to \mathbb{R}, \quad t \mapsto n + \tilde{g}(t).$$

Since $p \circ \tilde{\tilde{g}}(t) = p\big(n + \tilde{g}(t)\big) = p\big(\tilde{g}(t)\big)$, the path $\tilde{\tilde{g}}$ is then the lifting of g that begins at n. Then $\tilde{f} * \tilde{\tilde{g}} : [0,\, 1] \to \mathbb{R}$ is defined, and is the lifting of $f * g$ that begins at 0. As $\tilde{f} * \tilde{\tilde{g}}(1) = \tilde{\tilde{g}}(1) = n + m$, we obtain

$$\phi\big([f] * [g]\big) = n + m = \phi\big([f]\big) + \phi\big([g]\big).$$

□

Chapter 9

Homotopy

9.1 Homotopy of Functions

Definition 9.1. Let X,Y be topological spaces, and f,g continuous functions from X into Y. One says that f is **homotopic** to g if there is a continuous function $F : X \times [0,1] \to Y$ such that

$$\forall x \in X, \quad F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x).$$

In that case, one writes $f \simeq g$. The function F is called a **homotopy** between f and g.

Lemma 9.2. *The relation $\simeq$ on homotopic functions is an equivalence relation.*

Proof. Given a function f, the function $F(x,t) = f(x)$ is the required homotopy to get $f \simeq f$.
If $f \simeq g$ is got by a homotopy $F(x,t)$, then $G(x,t) = F(x, 1-t)$ is a homotopy between g and f.
Suppose that $f \simeq g$ by means of a homotopy F, and $g \simeq h$ by means of a homotopy G, then $f \simeq h$ by means of the homotopy $H : X \times [0,1] \to Y$ defined by the equation

$$H(x,t) = \begin{cases} F(x, 2t) & \text{if } t \in \left[0, \frac{1}{2}\right], \\ G(x, 2t-1) & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

□

Definition 9.3. Let X be a topological space, and $A \subseteq X$. A **retraction** of X onto A is a continuous function $r : X \to A$ such that the restriction $r : A \to A$ is the identity map of A. If such a function r exists, one says that A is a **retract** of X.

Definition 9.4. Let X be a topological space, and $A \subseteq X$. Suppose that there exists a continuous function $F : X \times [0,1] \to X$ such that

$$\forall x \in X, \quad F(x,0) = x \quad \text{and} \quad F(x,1) \in A,$$
$$\forall t \in [0,1],\ \forall a \in A, \quad F(a,t) = a.$$

The homotopy F between the identity map $F(x,0)$ of X and the retraction $F(x,1)$ of X onto A is called a **deformation retraction** of X onto A, and A is called a **deformation retract** of X.

Proposition 9.5. *Let X be a topological space, $A \subseteq X$, and $x \in A$. Consider the homomorphism $i_* : \pi_1(A,x) \to \pi_1(X,x)$ induced by the inclusion map $i : A \hookrightarrow X$.*

(i) If A is a retract of X, then i_* is injective.

(ii) If A is a deformation retract of X, then i_* is bijective.

Proof. (i) : If $r : X \to A$ is a retraction, then $r \circ i$ is the identity map of A. It follows that $(r \circ i)_* = r_* \circ i_*$ is the identity map of $\pi_1(A, x)$, which implies that i_* is injective.

(ii) : Suppose that $F : X \times [0,1] \to X$ is a deformation retraction of X onto A. Since $F(X,1) = A$, then for any loop $f : [0, 1] \to X$ based at x, $F\big(f(.), .\big)$ is a homotopy between f and a loop $F\big(f(.), 1\big)$ in A. Moreover, as $F\big(f(0), t\big) = F\big(f(1), t\big) = x$ for every $t \in [0, 1]$, then $f \simeq_p F\big(f(.), 1\big)$. Hence $\Big[F\big(f(.), 1\big)\Big] = [f]$, meaning that $[f] = i_*\bigg(\Big[F\big(f(.), 1\big)\Big]\bigg)$, and i_* is consequently surjective. □

Example. There is no retraction of th real disc $\overline{B\big((0,0), 1\big)}$ onto $\mathbb{S}^1$. Suppose, indeed, that $\mathbb{S}^1$ is a retract of $\overline{B\big((0,0), 1\big)}$. According to Proposition 9.5, the homomorphism $i_* : \pi_1\big(\mathbb{S}^1, (1,0)\big) \to \pi_1\Big(\overline{B\big((0,0), 1\big)}, (1,0)\Big)$ induced by the inclusion map $i : \mathbb{S}^1 \hookrightarrow \overline{B\big((0,0), 1\big)}$ is injective. That is impossible, since $\pi_1\big(\mathbb{S}^1, (1,0)\big) \cong \mathbb{Z}$ and $\pi_1\Big(\overline{B\big((0,0), 1\big)}, (1,0)\Big) \cong 0$.

9.2 Homotopy Equivalence

Definition 9.6. Let X, Y be a topological spaces, and $f : X \to Y$, $g : Y \to X$ continuous functions. Suppose that $g \circ f : X \to X$ is homotopic to the identity map of X, and $f \circ g : Y \to Y$ to the identity map of Y. Then, the functions f and g are said to be **homotopy equivalent**, and each is called a **homotopy inverse** of the other.

Proposition 9.7. *Let X, Y be topological spaces, and $F : X \times [0, 1] \to Y$ a homotopy between continuous functions $f = F(., 0)$ and $g = F(., 1)$. Take $x \in X$, and consider the path $h = F(x, .)$ from $f(x)$ to $g(x)$. Then, the following diagram is commutative:*

$$\begin{array}{ccc} \pi_1(X, x) & \xrightarrow{f_*} & \pi_1\big(Y, f(x)\big) \\ & \searrow^{g_*} & \downarrow \hat{h} \\ & & \pi_1\big(Y, g(x)\big) \end{array}$$

Proof. Let $l : [0, 1] \to X$ be a loop based at x. Consider the continuous function

$$L : [0, 1] \times [0, 1] \to Y, \quad (s,t) \mapsto F\big(l(s), t\big),$$

and the points $p_1 = (0, 0)$, $p_2 = (1, 0)$, $p_3 = (0, 1)$, $p_4 = (1, 1)$. Denoting L_{ij} the standard parametrisation of L restricted to the edge from p_i to p_j, where $i, j \in \{1,2,3,4\}$ and $i < j$, we get $L_{12} * L_{24} \simeq_p L_{14}$ and $L_{13} * L_{34} \simeq_p L_{14}$ by Lemma 7.16, hence $L_{12} * L_{24} \simeq_p L_{13} * L_{34}$. Remark that $L_{12} = f \circ l$, $L_{13} = L_{24} = h$, $L_{34} = g \circ l$, hence

$$\begin{aligned} f \circ l * h &= h * g \circ l \\ [f \circ l] * [h] &= [h] * [g \circ l] \\ [\bar{h}] * [f \circ l] * [h] &= [g \circ l] \\ \hat{h} \circ f_*\big([l]\big) &= g_*\big([l]\big). \end{aligned}$$

□

Corollary 9.8. *Let X be a topological space, and $f: X \to X$ a continuous function that is homotopic to the identity map of X. Then, for any $x \in X$, the function $f_*: \pi_1(X, x) \to \pi_1\big(X, f(x)\big)$ is a group isomorphism.*

Proof. Let $F: X \times [0, 1] \to X$ be a homotopy between the identity map $F(., 0) = i$ of X and $F(., 1) = f$, and consider the path $h = F(x, .)$ from x to $f(x)$. Proposition 9.7 implies that $f_* = \hat{h} \circ i_* = \hat{h}$, which is a isomorphism from $\pi_1(X, x)$ to $\pi_1\big(X, f(x)\big)$ by Proposition 7.11. $\square$

Lemma 9.9. *Let A, B, C, D be sets, and f, g, h functions represented by the following diagram:*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D.$$

If $g \circ f$ is bijective and $h \circ g$ is injective, then f is bijective.

Proof. As $g \circ f$ is injective, then f is injective.
Take $b \in B$. As $g \circ f$ is surjective, there exists $a \in A$ such that $g \circ f(a) = g(b)$. Remark that g is also injective since $h \circ g$ is injective. The injectivity of g implies $f(a) = b$, hence h is surjective. $\square$

Theorem 9.10. *Let X, Y be topological spaces, $x \in X$, and $f: X \to Y$ a continuous function. If there exists a continuous function $g: Y \to X$ homotopy equivalent to f, then $f_*: \pi_1(X, x) \to \pi_1\big(Y, f(x)\big)$ is an isomorphism.*

Proof. Consider the following sequence of homomorphisms:

$$\pi_1(X, x) \xrightarrow{f_*} \pi_1\big(Y, f(x)\big) \xrightarrow{g_*} \pi_1\big(X, g \circ f(x)\big) \xrightarrow{f_*} \pi_1\big(Y, f \circ g \circ f(x)\big).$$

We know from Corollary 9.8 that $g_* \circ f_*$ and $f_* \circ g_*$ are isomorphisms. Morevore, we can deduce from Lemma 9.9 that f_* is bijective. $\square$

Chapter 10

Singular Homology

10.1 Singular Homology

Proposition 10.1. *Let $u_0, u_1, \ldots, u_p \in \mathbb{R}^n$. The following conditions are equivalent:*

(i) *the p vectors $\overrightarrow{u_0u_1}, \overrightarrow{u_0u_2}, \ldots \overrightarrow{u_0u_p}$ are linearly independent,*

(ii) *if $s_0, s_1, \ldots, s_p, t_0, t_1, \ldots, t_p \in \mathbb{R}$ such that*

$$\sum_{i=0}^{p} s_i u_i = \sum_{i=0}^{p} t_i u_i \quad \textit{and} \quad \sum_{i=0}^{p} s_i = \sum_{i=0}^{p} t_i,$$

then $s_i = t_i$ for $i \in \{0, 1, \ldots, p\}$.

Proof. $(i) \Rightarrow (ii)$: If $\sum_{i=0}^{p} s_i u_i = \sum_{i=0}^{p} t_i u_i$ and $\sum_{i=0}^{p} s_i = \sum_{i=0}^{p} t_i$, then

$$\begin{aligned} 0 &= \sum_{i=0}^{p} (s_i - t_i) u_i \\ &= \sum_{i=0}^{p} (s_i - t_i) u_i - \left(\sum_{i=0}^{p} (s_i - t_i) \right) u_0 \\ &= \sum_{i=1}^{p} (s_i - t_i)(u_i - u_0). \end{aligned}$$

As $\overrightarrow{u_0u_1}, \overrightarrow{u_0u_2}, \ldots \overrightarrow{u_0u_p}$ are linearly independent, it follows that $s_i = t_i$ for $i \in \{1, \ldots, p\}$. Moreover, $\sum_{i=0}^{p} s_i = \sum_{i=0}^{p} t_i$ implies $s_0 = t_0$.

$(ii) \Rightarrow (i)$: If $\sum_{i=1}^{p} t_i(u_i - u_0) = 0$, then $\sum_{i=1}^{p} t_i u_i = \left(\sum_{i=1}^{p} t_i \right) u_0$. Hence, we must have $t_1 = \cdots = t_n = 0$. □

Definition 10.2. Let $n \in \mathbb{N}$, $p \in \{1, \ldots, n\}$, and $u_0, u_1, \ldots, u_p \in \mathbb{R}^n$. A ***p*-simplex** $[u_0, u_1, \ldots, u_p]$ is a convex hull

$$\left\{ t_0 u_0 + t_1 u_1 + \cdots + t_p u_p \;\middle|\; t_0, t_1, \ldots, t_p \in \mathbb{R}_+, \sum_{i=0}^{p} t_i = 1 \right\}$$

with ordered **vertices** $u_0, u_1, \ldots, u_p$ such that the p vectors $\overrightarrow{u_0u_1}, \overrightarrow{u_0u_2}, \ldots \overrightarrow{u_0u_p}$ are linearly independent.

Corollary 10.3. *If $[u_0,u_1,\dots,u_p]$ is a p-simplex in $\mathbb{R}^n$, then every point of $[u_0,u_1,\dots,u_p]$ has a distinct unique representation in the form* $\sum_{i=0}^{p} t_iu_i$, *with* $t_0,t_1,\dots,t_p \in \mathbb{R}_+$ *and* $\sum_{i=0}^{p} t_i = 1$.

Proof. It is Proposition 10.1 with the conditions $t_0,t_1,\dots,t_p \in \mathbb{R}_+$ and $\sum_{i=0}^{p} t_i = 1$. □

Example. The **standard n-simplex** is convex hull

$$\Delta^n := \Big\{(t_0,t_1,\dots,t_n) \in \mathbb{R}^{n+1} \,\Big|\, t_0,t_1,\dots,t_p \in \mathbb{R}_+, \sum_{i=0}^{n} t_i = 1\Big\} = [e_0,e_1,\dots,e_n]$$

of the ordered vertices $e_0 = (0,\dots,0)$, $e_1 = (1,0,\dots,0)$, $\dots$, $e_n = (0,\dots,0,1)$.

Definition 10.4. Let X be a topological space. A **singular n-simplex** in X is a continuous function

$$\sigma : \Delta^n \to X.$$

Denote $S_n(X)$ the set of singular n-simplices in X. Let $C_n(X)$ be the free abelian group with basis $S_n(X)$, that is,

$$C_n(X) := \Big\{ \sum_{a\in A} n_a\sigma_a \,\Big|\, \#A \in \mathbb{N},\, n_a \in \mathbb{Z},\, \sigma_a \in S_n(X)\Big\}.$$

Elements of $C_n(X)$ are called **singular n-chains**.

Definition 10.5. Let X be a topological space, and $i \in \{0,1,\dots,n\}$. The i^{th} **face operator** is the homomorphism

$$\partial_i : C_n(X) \to C_{n-1}(X), \quad \sum_{a\in A} n_a\sigma_a \mapsto \sum_{a\in A} n_a\sigma_a|[e_0,e_1,\dots,\hat{e}_i,\dots,e_n],$$

where $[e_0,e_1,\dots,\hat{e}_i,\dots,e_n]$ is the $n-1$-simplex with vertices $e_0,\dots,e_{i-1},e_{i+1},\dots,e_n$.
The **boundary operator** is the homomorphism

$$\partial : C_n(X) \to C_{n-1}(X), \quad \sigma \mapsto \sum_{i=0}^{n} (-1)^i \partial_i(\sigma).$$

Proposition 10.6. *Let X be a topological space. The following composition is zero:*

$$C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} C_{n-2}(X).$$

Proof. For $\sigma \in C_n(X)$, we have $\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|[e_0,\dots,\hat{e}_i,\dots,e_n]$. Remark that

$$\partial\sigma|[e_0,\dots,\hat{e}_i,\dots,e_n] = \sum_{j=0}^{i-1} (-1)^j \sigma|[e_0,\dots,\hat{e}_j,\dots,\hat{e}_i,\dots,e_n] + \sum_{j=i+1}^{n} (-1)^{j-1} \sigma|[e_0,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_n].$$

Then,

$$\begin{aligned}
\partial\circ\partial(\sigma) &= \sum_{i=0}^{n}\sum_{j=0}^{i-1} (-1)^{i+j} \sigma|[e_0,\dots,\hat{e}_j,\dots,\hat{e}_i,\dots,e_n] + \sum_{i=0}^{n}\sum_{j=i+1}^{n} (-1)^{i+j-1} \sigma|[e_0,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_n] \\
&= \sum_{\substack{i,j\in\{0,\dots,n\} \\ i>j}} (-1)^{i+j} \sigma|[e_0,\dots,\hat{e}_j,\dots,\hat{e}_i,\dots,e_n] + \sum_{\substack{i,j\in\{0,\dots,n\} \\ i<j}} (-1)^{i+j-1} \sigma|[e_0,\dots,\hat{e}_i,\dots,\hat{e}_j,\dots,e_n] \\
&= 0.
\end{aligned}$$

□

Definition 10.7. Let X be a topological space. The **singular complex** $C_\bullet(X)$ of X is the homomorphism sequence

$$\cdots \xrightarrow{\partial} C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\partial} 0.$$

The group of **singular** n**-cycles** of X is $Z_n(X) := \big\{\sigma \in C_n(X) \;\big|\; \partial(\sigma) = 0\big\}$. The group of **singular** n**-boundaries** of X is $B_n(X) := \big\{\sigma \in C_n(X) \;\big|\; \exists \tau \in C_{n+1}(X),\, \partial(\tau) = \sigma\big\}$. The quotient group

$$H_n(X) = Z_n(X)/B_n(X)$$

is the n^{th} **singular homology group** of X.

Example. If x is a point, then $H_0\big(\{x\}\big) \cong \mathbb{Z}$, and $H_n\big(\{x\}\big) = 0$ for $n \in \mathbb{N}$. Indeed, for every nonnegative integer n, $C_n\big(\{x\}\big) = \mathbb{Z}\{\sigma\}$, where $\sigma : \Delta^n \to \{x\}$, $t \mapsto x$. Moreover, for every $z\sigma \in C_n\big(\{x\}\big)$,

$$\partial(z\sigma) = \sum_{i=0}^{n} (-1)^i \partial_i(z\sigma) = \sum_{i=0}^{n} (-1)^i z\sigma = \begin{cases} z\sigma & \text{if } n \text{ is even and } n \neq 0, \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

The singular complex of $\{x\}$ is then

$$\cdots \xrightarrow{0} \mathbb{Z}\{\sigma\} \xrightarrow{\text{restriction}} \mathbb{Z}\{\sigma\} \xrightarrow{0} \mathbb{Z}\{\sigma\} \xrightarrow{\text{restriction}} \mathbb{Z}\{\sigma\} \xrightarrow{0} \mathbb{Z}\{\sigma\} \xrightarrow{0} 0.$$

Hence,

- $Z_0\big(\{x\}\big) = \mathbb{Z}\{\sigma\}$ and $B_0\big(\{x\}\big) = \{0\}$, implying $H_0\big(\{x\}\big) = \mathbb{Z}\{\sigma\}/\{0\} \cong \mathbb{Z}$,
- if n is even and $n \neq 0$, $Z_n\big(\{x\}\big) = \{0\}$ and $B_n\big(\{x\}\big) = \{0\}$, then $H_n\big(\{x\}\big) = \{0\}/\{0\} = \{0\}$,
- if n is odd, $Z_n\big(\{x\}\big) = \mathbb{Z}\{\sigma\}$ and $B_n\big(\{x\}\big) = \mathbb{Z}\{\sigma\}$, then $H_n\big(\{x\}\big) = \mathbb{Z}\{\sigma\}/\mathbb{Z}\{\sigma\} \cong \{0\}$.

Proposition 10.8. *Let X be a topological space. Suppose that $X = \bigsqcup_{i \in I} X_i$, where X_i is a path component. Then,*

$$H_n(X) \cong \bigoplus_{i \in I} H_n(X_i).$$

Proof. Let σ be a singular n-simplex in X. Since Δ^n is path connected, then $\sigma(\Delta^n)$ is path connected, meaning that $\sigma(\Delta^n) \subseteq X_i$ for some $i \in I$. Then $C_n(X) = \bigoplus_{i \in I} C_n(X_i)$. Moreover, $\partial\big(C_n(X_i)\big) \subseteq C_{n-1}(X_i)$, hence $Z_n(X) = \bigoplus_{i \in I} Z_n(X_i)$ and $B_n(X) = \bigoplus_{i \in I} B_n(X_i)$. Consider the natural homomorphism $p : \bigoplus_{i \in I} Z_n(X_i) \mapsto \bigoplus_{i \in I} Z_n(X_i)/B_n(X_i)$, $(\sigma_i)_{i \in I} \mapsto (\dot{\sigma}_i)_{i \in I}$ which the canonical projection on each coordinate. It is obviously surjective, and $\ker p = \bigoplus_{i \in I} B_n(X_i)$. Therefore

$$H_n(X) = \bigoplus_{i \in I} Z_n(X_i) \Big/ \bigoplus_{i \in I} B_n(X_i) \cong \bigoplus_{i \in I} Z_n(X_i)/B_n(X_i) = \bigoplus_{i \in I} H_n(X_i).$$

□

Proposition 10.9. *Let X be a topological space. Suppose that $X = \bigsqcup_{i \in I} X_i$, where X_i is a path component. Then,*

$$H_0(X) \cong \overbrace{\cdots \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots}^{\#I \text{ times}}.$$

Proof. Define a homomorphism $h : C_0(X_i) \to \mathbb{Z},\ \sum_{j \in J} n_j \sigma_j \mapsto \sum_{j \in J} n_j$. It is obviously surjective as X_i is assumed to be nonempty. For every $\sigma \in S_1(X_i)$, we have $h \circ \partial(\sigma) = h\big(\sigma|[e_1] - \sigma|[e_0]\big) = 1 - 1 = 0$. It follows that $\big\{\tau \in C_0(X_i) \,\big|\, \exists \sigma \in C_1(X_i),\, \partial(\sigma) = \tau\big\} = B_0(X_i) \subseteq \ker h$.
Now, let $\sum_{j \in J} n_j \sigma_j \in C_0(X_i)$ such that $h\Big(\sum_{j \in J} n_j \sigma_j\Big) = 0$. Take a point $x \in X_i$ and note that, for each $j \in J$, there exists a singular 1-simplex $\tau_j : [e_0, e_1] \to X_i$ such that $\tau_j(e_0) = \sigma(e_0)$ and $\tau_j(e_1) = x$. We have

$$\partial\Big(\sum_{j \in J} n_j \tau_j\Big) = \sum_{j \in J} n_j \sigma_j - \Big(\sum_{j \in J} n_j\Big)\phi = \sum_{j \in J} n_j \sigma_j \quad \text{with} \quad \phi : [e_0] \to X_i,\ e_0 \mapsto x.$$

Hence $\ker h \subseteq \big\{\sigma \in C_0(X_i) \,\big|\, \exists \tau \in C_1(X_i),\, \partial(\tau) = \sigma\big\} = B_0(X_i)$.
We deduce that $B_0(X_i) = \ker h$. Therefore

$$H_0(X_i) = Z_0(X_i)/B_0(X_i) = C_0(X_i)/\ker h \cong h\big(C_0(X_i)\big) = \mathbb{Z}.$$

Finally, we get $H_0(X) \cong \overbrace{\cdots \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots}^{\#I \text{ times}}$ by Proposition 10.8. □

10.2 Homotopy Invariance

Definition 10.10. Let X, Y be topological spaces, and $f : X \to Y$ a continuous function. The **homomorphism induced on singular n-chains** by f is

$$f_\sharp : C_n(X) \to C_n(Y), \quad \sum_{a \in A} n_a \sigma_a \mapsto \sum_{a \in A} n_a f \circ \sigma_a.$$

Lemma 10.11. *Let X, Y be topological spaces, and $f : X \to Y$ a continuous function. The following diagram is commutative:*

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial} & C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) & \xrightarrow{\partial} & \cdots \\ & & \downarrow f_\sharp & & \downarrow f_\sharp & & \downarrow f_\sharp & & \\ \cdots & \xrightarrow{\partial} & C_{n+1}(Y) & \xrightarrow{\partial} & C_n(Y) & \xrightarrow{\partial} & C_{n-1}(Y) & \xrightarrow{\partial} & \cdots \end{array}$$

Proof. Let $\sigma \in C_n(X)$. We have

$$\begin{aligned} f_\sharp \circ \partial(\sigma) &= f_\sharp\Big(\sum_{i=0}^{n} (-1)^i \sigma|[e_0, e_1, \ldots, \hat{e}_i, \ldots, e_n]\Big) \\ &= \sum_{i=0}^{n} (-1)^i f_\sharp \circ \sigma|[e_0, e_1, \ldots, \hat{e}_i, \ldots, e_n] \\ &= \partial(f_\sharp \circ \sigma). \end{aligned}$$

□

Proposition 10.12. *Let X, Y be topological spaces, and $f : X \to Y$ a continuous function. Then, $f_\sharp$ induces a homomorphism*

$$f_\star : H_n(X) \to H_n(Y), \quad \sigma + B_n(X) \mapsto f_\sharp(\sigma) + B_n(Y).$$

Proof. Using Lemma 10.11:

- If $\sigma \in Z_n(X)$, then $\partial\big(f_\sharp(\sigma)\big) = f_\sharp\big(\partial(\sigma)\big) = f_\sharp(0) = 0$, so $f_\sharp\big(Z_n(X)\big) \subseteq Z_n(Y)$,
- if $\sigma \in C_{n+1}(X)$, then $f_\sharp\big(\partial(\sigma)\big) = \partial\big(f_\sharp(\sigma)\big)$, so $f_\sharp\big(B_n(X)\big) \subseteq B_n(Y)$.

Hence, for every $\sigma + B_n(X) \in H_n(X)$, $f_\star\big(\sigma + B_n(X)\big) = f_\sharp(\sigma) + B_n(Y) \in H_n(Y)$ is well-defined. And $f_\star\big(\sigma + \tau + B_n(X)\big) = f_\sharp(\sigma+\tau) + B_n(Y) = f_\sharp(\sigma) + f_\sharp(\tau) + B_n(Y) = f_\star\big(\sigma + B_n(X)\big) + f_\star\big(\tau + B_n(X)\big)$. ☐

Definition 10.13. Let X, Y be topological spaces, and $f : X \to Y$ a continuous function. The **homomorphism induced on homology groups** by f is

$$f_\star : H_n(X) \to H_n(Y), \quad \sigma + B_n(X) \mapsto f_\sharp(\sigma) + B_n(Y).$$

Proposition 10.14. *Let X, Y, Z be topological spaces, and $f : X \to Y$, $g : Y \to Z$ continuous functions. In particular, let $i_X : X \to X$ and $i : H_n(X) \to H_n(X)$ be the identity maps of X and $H_n(X)$ respectively. Then,*

(i) $(g \circ f)_\star = g_\star \circ f_\star$,

(ii) $(i_X)_\star = i$.

Proof. (i) : If $\sum_{a \in A} n_a \sigma_a \in C_n(X)$, we have

$$g_\sharp \circ f_\sharp\Big(\sum_{a \in A} n_a \sigma_a\Big) = g_\sharp\Big(\sum_{a \in A} n_a f \circ \sigma_a\Big) = \sum_{a \in A} n_a g \circ f \circ \sigma_a = (g \circ f)_\sharp\Big(\sum_{a \in A} n_a \sigma_a\Big).$$

Hence, if $\sigma + B_n(X) \in H_n(X)$,

$$\begin{aligned} g_\star \circ f_\star\big(\sigma + B_n(X)\big) &= g_\star\big(f_\sharp(\sigma) + B_n(Y)\big) \\ &= g_\sharp \circ f_\sharp(\sigma) + B_n(Z) \\ &= (g \circ f)_\sharp(\sigma) + B_n(Z) \\ &= (g \circ f)_\star\big(\sigma + B_n(X)\big). \end{aligned}$$

(ii) : For $\sigma + B_n(X) \in H_n(X)$, $(i_X)_\star\big(\sigma + B_n(X)\big) = (i_X)_\sharp(\sigma) + B_n(X) = \sigma + B_n(X)$. ☐

For a nonnegative integer n, set $\Delta^n \times \{0\} := [e_0^0, e_1^0, \dots, e_n^0]$ and $\Delta^n \times \{1\} := [e_0^1, e_1^1, \dots, e_n^1]$ such that e_i^0 and e_i^1 have the same image e_i under the projection $\Delta^n \times \{0, 1\} \to \Delta^n$, where $i \in \{0, 1, \dots, n\}$.

Proposition 10.15. *Let n be a nonnegative integer. Then*

$$\Delta^n \times [0, 1] = \bigcup_{i=0}^{n} [e_0^0, \dots, e_{i-1}^0, e_i^0, e_i^1, e_{i+1}^1, \dots, e_n^1].$$

Proof. Let $u = \sum_{j=0}^{i} t_j^0 e_j^0 + \sum_{j=i}^{n} t_j^1 e_j^1 \in [e_0^0, \dots, e_{i-1}^0, e_i^0, e_i^1, e_{i+1}^1, \dots, e_n^1]$. If $u = (\lambda_0, \lambda_1, \dots, \lambda_{n+1})$, then

$$\sum_{k=0}^{n} \lambda_k = \sum_{j=0}^{i} t_j^0 + \sum_{j=i}^{n} t_j^1 = 1 \quad \text{and} \quad \lambda_{n+1} = \sum_{j=i}^{n} t_j^1 \in [0,\, 1].$$

Hence $[e_0^0, \dots, e_{i-1}^0, e_i^0, e_i^1, e_{i+1}^1, \dots, e_n^1] \subseteq \Delta^n \times [0,\, 1]$.
Now, take $(\lambda_0, \lambda_1, \dots, \lambda_{n+1}) \in \Delta^n \times [0,\, 1]$. Let $i = \max\Big\{ j \in \{0, 1, \dots, n\} \;\Big|\; \sum_{j=i}^{n} \lambda_j \geq \lambda_{n+1} \Big\}$. Then,

$$(\lambda_0, \lambda_1, \dots, \lambda_{n+1}) = \sum_{j=0}^{i-1} \lambda_j e_j^0 + \Big(\lambda_i - \lambda_{n+1} + \sum_{j=i}^{n} \lambda_j\Big) e_i^0 + \Big(\lambda_{n+1} - \sum_{j=i}^{n} \lambda_j\Big) e_i^1 + \sum_{j=i+1}^{n} \lambda_j e_j^1$$

which belongs to $[e_0^0, \dots, e_i^0, e_i^1, \dots, e_n^1]$. Hence $\Delta^n \times [0,\, 1] \subseteq \bigcup_{i=0}^{n} [e_0^0, \dots, e_i^0, e_i^1, \dots, e_n^1]$. ☐

Definition 10.16. Let X, Y be topological spaces, $id : [0,\, 1] \to [0,\, 1]$ the identity map, and $F : X \times [0,\, 1] \to Y$ a continuous function. The composition $F \circ (\sigma \times id) : \Delta^n \times [0,\, 1] \to X \times [0,\, 1] \to Y$ is well-defined and the **prism operator** of F is the function

$$P : C_n(X) \to C_{n+1}(Y), \quad \sigma \mapsto \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times id)|[e_0^0, \dots, e_{i-1}^0, e_i^0, e_i^1, e_{i+1}^1, \dots, e_n^1].$$

Proposition 10.17. *Let X, Y be topological spaces, $f : X \to Y$, $g : X \to Y$ continuous functions, and $F : X \times [0,\, 1] \to Y$ a homotopy between f and g. Then,*

$$\partial \circ P = g_\sharp - f_\sharp - P \circ \partial.$$

Proof. Denote

$$F_{i,j}^0 = F \circ (\sigma \times id)|[e_0^0, \dots, \widehat{e_j^0}, \dots, e_i^0, e_i^1, \dots, e_n^1] \text{ and } F_{i,j}^1 = F \circ (\sigma \times id)|[e_0^0, \dots, e_i^0, e_i^1, \dots, \widehat{e_j^1}, \dots, e_n^1].$$

We have

$$\begin{aligned}
\partial \circ P(\sigma) &= \partial\Big(\sum_{i=0}^{n} (-1)^i F \circ (\sigma \times id)|[e_0^0, \dots, e_{i-1}^0, e_i^0, e_i^1, e_{i+1}^1, \dots, e_n^1]\Big) \\
&= \sum_{i=0}^{n} (-1)^i \partial\Big(F \circ (\sigma \times id)|[e_0^0, \dots, e_{i-1}^0, e_i^0, e_i^1, e_{i+1}^1, \dots, e_n^1]\Big) \\
&= \sum_{i=0}^{n} (-1)^i \Big(\sum_{j=0}^{i} (-1)^j F_{i,j}^0 + \sum_{j=i}^{n} (-1)^{j+1} F_{i,j}^1\Big) \\
&= \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{i+j} F_{i,j}^0 + \sum_{i=0}^{n} \sum_{j=i}^{n} (-1)^{i+j+1} F_{i,j}^1
\end{aligned}$$

Remark that $[e_0^0, \dots, e_i^0, \widehat{e_i^1}, e_{i+1}^1, \dots, e_n^1] = [e_0^0, \dots, e_i^0, \widehat{e_{i+1}^0}, e_{i+1}^1, \dots, e_n^1]$, which implies $F_{i,i}^1 = F_{i+1,i+1}^0$. Hence

$$\partial \circ P(\sigma) = F_{0,0}^0 + \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^{i+j} F_{i,j}^0 + \sum_{i=0}^{n} \sum_{j=i+1}^{n} (-1)^{i+j+1} F_{i,j}^1 - F_{n,n}^1.$$

Note that $F^0_{0,0} = F\circ(\sigma\times id)|[\widehat{e^0_0}, e^1_0, e^1_1, \dots, e^1_n] = g_\sharp$ and $F^1_{n,n} = F\circ(\sigma\times i)|[e^0_0,\dots,e^0_{n-1}, e^0_n, \widehat{e^1_n}] = f_\sharp$. Moreover,

$$\begin{aligned}P\circ\partial(\sigma) &= P\Big(\sum_{i=0}^{n}(-1)^i\sigma|[e_0,\dots,\hat{e}_i,\dots,e_n]\Big)\\ &= \sum_{i=0}^{n}(-1)^i\sum_{j=i+1}^{n}(-1)^jF^1_{i,j} + \sum_{i=0}^{n}(-1)^{i-1}\sum_{j=0}^{i-1}(-1)^jF^0_{i,j}\\ &= \sum_{i=0}^{n}\sum_{j=0}^{i-1}(-1)^{i+j-1}F^0_{i,j} + \sum_{i=0}^{n}\sum_{j=i+1}^{n}(-1)^{i+j}F^1_{i,j}.\end{aligned}$$

Therefore $\partial\circ P = g_\sharp - P\circ\partial - f_\sharp$. □

Theorem 10.18. *Let X, Y be topological spaces, and $f : X\to Y$, $g : X\to Y$ continuous functions. If f and g are homotopic, then $f_\star = g_\star$.*

Proof. Let P be the prism operator of a homotopy between f and g. If $\sigma\in Z_n(X)$, we then know from Proposition 10.17 that $g_\sharp(\sigma) - f_\sharp(\sigma) = \partial\circ P(\sigma) + P\circ\partial(\sigma) = \partial\circ P(\sigma)$, since $\partial(\sigma) = 0$. Thus $g_\sharp(\sigma) - f_\sharp(\sigma)\in B_n(Y)$, meaning that $g_\sharp(\sigma) + B_n(Y) = f_\sharp(\sigma) + B_n(Y)$. So, for all $\sigma + B_n(X)\in H_n(X)$,

$$g_\star\big(\sigma + B_n(X)\big) = g_\sharp(\sigma) + B_n(Y) = f_\sharp(\sigma) + B_n(Y) = f_\star\big(\sigma + B_n(X)\big).$$

□

Corollary 10.19. *Let X, Y be topological spaces, and $f : X\to Y$ a continuous function. If f is homotopy equivalent some function, then $f_\star : H_n(X)\to H_n(Y)$ is an isomorphism.*

Proof. Let $g : Y\to X$ be a function homotopy equivalent to f. Moreover, let $i_X, i_Y, i_{H_n(X)}, i_{H_n(Y)}$ be the identity maps of $X, Y, H_n(X), H_n(Y)$ respectively. Using Proposition 10.14 and Theorem 10.18, we get

- $g_\star\circ f_\star = (g\circ f)_\star = (i_X)_\star = i_{H_n(X)}$,
- $f_\star\circ g_\star = (f\circ g)_\star = (i_Y)_\star = i_{H_n(Y)}$.

Hence, $g_\star = f_\star^{-1}$, which implies that $f_\star$ is an isomorphism. □

Example. If X is a convex set in $\mathbb{R}^n$, then $H_0(X)\cong\mathbb{Z}$, and $H_n(X) = 0$ for $n\in\mathbb{N}$. Indeed, if $a\in X$, the function

$$F : X\times[0,1]\to X,\quad (x,t)\mapsto ta + (1-t)x$$

is a deformation retraction of X onto $\{a\}$. Consider both functions

$$f : X\to\{a\},\ x\mapsto a \quad\text{and}\quad g : \{a\}\to X,\ x\to x.$$

Denoting $i_X, i_{\{a\}}$ the identity maps of X and $\{a\}$ respectively, we see that

- $g\circ f = f$ which is homotopic to i_X by the deformation retraction F,
- $f\circ g = i_{\{a\}}$.

Hence, f and g are homotopy equivalent. We deduce from Corollary 10.19 that $f_\star : H_n(X)\to H_n\big(\{a\}\big)$ is an isomorphism.

10.3 Relative Homology Groups

Definition 10.20. Let X be a topological space, and $A \subseteq X$. The free abelian subgroup $C_n(A)$ is

$$C_n(A) := \Big\{ \sum_{i \in I} n_i \sigma_i \in C_n(X) \;\Big|\; \sigma_i(\Delta^n) \subseteq A \Big\}.$$

The **relative n-chains** are the elements of the quotient group $C_n(X, A) := C_n(X)/C_n(A)$.

Lemma 10.21. *Let X be a topological space, and $A \subseteq X$. The boundary operator $\partial : C_n(X) \to C_{n-1}(X)$ induces the quotient boundary operator*

$$\dot{\partial} : C_n(X, A) \to C_{n-1}(X, A), \quad \sigma + C_n(A) \mapsto \dot{\partial}(\sigma) + C_{n-1}(A).$$

Proof. Let $\tau = \sum_{i \in I} n_i \tau_i \in C_n(A)$ and $j \in \{0, 1, \dots, n\}$. Since $\tau_i|[e_0, e_1, \dots, \hat{e}_j, \dots, e_n](\Delta^{n-1}) \subseteq A$, then $\partial(\tau) \in C_{n-1}(A)$. Hence $\partial\big(C_n(A)\big) \subseteq C_{n-1}(A)$, and $\dot{\partial} : C_n(X, A) \to C_{n-1}(X, A)$ is well-defined. □

Definition 10.22. Let X be a topological space, and $A \subseteq X$. The **relative complex $C_\bullet(X, A)$** of X relative to A is

$$\cdots \xrightarrow{\dot{\partial}} C_{n+1}(X, A) \xrightarrow{\dot{\partial}} C_n(X, A) \xrightarrow{\dot{\partial}} C_{n-1}(X, A) \xrightarrow{\dot{\partial}} \cdots \xrightarrow{\dot{\partial}} C_1(X, A) \xrightarrow{\dot{\partial}} C_0(X, A) \xrightarrow{\dot{\partial}} 0.$$

The group of **relative n-cycles** of X relative to A is

$$Z_n(X, A) := \big\{ \sigma + C_n(A) \in C_n(X, A) \;\big|\; \partial(\sigma) \in C_{n-1}(A) \big\}.$$

The group of **relative n-boundaries** of X relative to A is

$$B_n(X, A) := \big\{ \sigma + C_n(A) \in C_n(X, A) \;\big|\; \exists \tau \in C_{n+1}(X),\, \upsilon \in C_n(A),\, \partial(\tau) = \sigma + \upsilon \big\}.$$

The quotient group

$$H_n(X, A) = Z_n(X, A)/B_n(X, A)$$

is the n^{th} **relative homology group** of X relative to A.

Denote $f : (X, A) \to (Y, B)$ a function $f : X \to Y$ such that $A \subseteq X$, $B \subseteq Y$, and $f(A) \subseteq B$.

Lemma 10.23. *Let X, Y be topological spaces, $A \subseteq X$, $B \subseteq Y$, and $f : (X, A) \to (Y, B)$ a continuous function. The homomorphism $f_\sharp : C_n(X) \to C_n(Y)$ induces the homomorphism on relative n-chains*

$$\dot{f}_\sharp : C_n(X, A) \to C_n(Y, B), \quad \sigma + C_n(A) \mapsto f_\sharp(\sigma) + C_n(B).$$

Proof. If $\sum_{i \in I} n_i \sigma_i \in C_n(A)$, then $f_\sharp\Big(\sum_{i \in I} n_i \sigma_i\Big) = \sum_{i \in I} n_i f \circ \sigma_i \in C_n(B)$. Hence $f_\sharp\big(C_n(A)\big) \subseteq C_n(B)$, and $\dot{f}_\sharp : C_n(X, A) \to C_n(Y, B)$ is well-defined. □

Lemma 10.24. *Let X, Y be topological spaces, $A \subseteq X$, $B \subseteq Y$, and $f : (X, A) \to (Y, B)$ a continuous function. The homomorphism $f_\star : H_n(X) \to H_n(Y)$ induces the homomorphism on relative homology groups*

$$\dot{f}_\star : H_n(X, A) \to H_n(Y, B), \quad \sigma + B_n(X, A) \mapsto f_\sharp(\sigma) + B_n(Y, B).$$

Proof. We have:

- If $\sigma + C_n(A) \in Z_n(X,A)$, then

$$\partial\Big(f_\sharp\big(\sigma + C_n(A)\big)\Big) = \partial\big(f_\sharp(\sigma) + C_n(B)\big) = \partial\big(f_\sharp(\sigma)\big) + \partial\big(C_n(B)\big) = f_\sharp\big(\partial(\sigma)\big) + \partial\big(C_n(B)\big).$$

  Since $\partial(\sigma) \in C_{n-1}(A)$, then $f_\sharp\big(\partial(\sigma)\big) + \partial\big(C_n(B)\big) \subseteq C_{n-1}(B)$, so $f_\sharp\big(Z_n(X,A)\big) \subseteq Z_n(Y,B)$.

- If $\sigma + C_{n+1}(A) \in C_{n+1}(X,A)$, then

$$f_\sharp\Big(\partial\big(\sigma + C_{n+1}(A)\big)\Big) = \partial\Big(f_\sharp\big(\sigma + C_{n+1}(A)\big)\Big) = \partial\big(f_\sharp(\sigma) + C_{n+1}(B)\big),$$

  hence $f_\sharp\big(B_n(X,A)\big) \subseteq B_n(Y,B)$.

Like in Proposition 10.12, we deduce that $f_\sharp$ induces a homomorphism $f_\star : H_n(X,A) \to H_n(Y,B)$. $\square$

Proposition 10.25. *Let X,Y be topological spaces, $A \subseteq X$, $B \subseteq Y$, and $f : (X,A) \to (Y,B)$, $g : (X,A) \to (Y,B)$ continuous functions. Suppose that there exists a homotopy $F : X \times [0,\,1] \to Y$ between f and g such that*

$$\forall t \in [0,\,1],\ F(A,t) \subseteq B.$$

Then $\dot{f}_\star : H_n(X,A) \to H_n(Y,B) = \dot{g}_\star : H_n(X,A) \to H_n(Y,B)$.

Proof. If $\sigma \in C_n(X)$ such that $\sigma(\Delta^n) \subseteq A$, we get the composition $F \circ (\sigma \times id) : \Delta^n \times [0,\,1] \to A \times [0,\,1] \to B$. The prism operator P of F then takes $C_n(A)$ to $C_{n+1}(B)$. Hence, it induces a relative prism operator

$$\dot{P} : C_n(X,A) \to C_{n+1}(Y,B), \quad \sigma + C_n(A) \mapsto P(\sigma) + C_{n+1}(B).$$

Besides, for every $\sigma + C_n(A) \in C_n(X,A)$, $\dot{\partial} \circ \dot{P}\big(\sigma + C_n(A)\big) = \dot{\partial}\big(P(\sigma) + C_{n+1}(B)\big) = \partial \circ P(\sigma) + C_n(B)$ and $\dot{P} \circ \dot{\partial}\big(\sigma + C_n(A)\big) = \dot{P}\big(\partial(\sigma) + C_{n-1}(A)\big) = P \circ \partial(\sigma) + C_n(B)$. So, by Proposition 10.17,

$$\begin{aligned} \dot{\partial} \circ \dot{P}\big(\sigma + C_n(A)\big) + \dot{P} \circ \dot{\partial}\big(\sigma + C_n(A)\big) &= \partial \circ P(\sigma) + P \circ \partial(\sigma) + C_n(B) \\ &= g_\sharp(\sigma) - f_\sharp(\sigma) + C_n(B) \\ &= \dot{g}_\sharp\big(\sigma + C_n(A)\big) - \dot{f}_\sharp\big(\sigma + C_n(A)\big). \end{aligned}$$

If $\sigma + C_n(A) \in Z_n(X,A)$, since $\dot{\partial}\big(\sigma + C_n(A)\big) = C_{n-1}(A)$, then

$$\dot{g}_\sharp\big(\sigma + C_n(A)\big) - \dot{f}_\sharp\big(\sigma + C_n(A)\big) = \dot{\partial} \circ \dot{P}\big(\sigma + C_n(A)\big).$$

Thus $\dot{g}_\sharp\big(\sigma + C_n(A)\big) - \dot{f}_\sharp\big(\sigma + C_n(A)\big) \in B_n(Y,B)$, meaning that $g_\sharp(\sigma) + B_n(Y,B) = f_\sharp(\sigma) + B_n(Y,B)$. So, for all $\sigma + B_n(X,A) \in H_n(X,A)$,

$$\dot{g}_\star\big(\sigma + B_n(X,A)\big) = g_\sharp(\sigma) + B_n(Y,B) = f_\sharp(\sigma) + B_n(Y,B) = \dot{f}_\star\big(\sigma + B_n(X,A)\big).$$

$\square$

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